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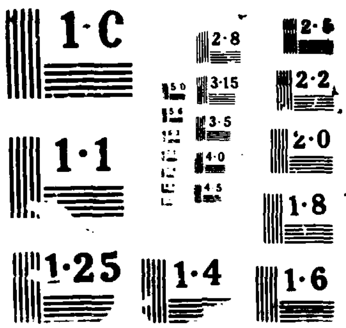
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ELECTROMAGNETIC SHADOWING BY AN EXPONENTIALLY DISTRIBUTED ROUGH SURFACE

THESIS

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THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

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Requirements for the Degree of

Master of Science in Electrical Engineering

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Preface

The purpose of this study was to obtain and analyze the electromagnetic shadowing function for a random rough surface described by a true exponential probability density function. It has been determined that large errors may occur if shadowing is neglected when computing scattering coefficients. This report developed and compared electromagnetic shadowing functions for three different probability density functions.

This project employed a unique blend of electromagnetic scattering, probability and advanced calculus principles.

I would like to thank my advisor, Dr. Vital Pyati, who derived the true exponential probability density function used in this thesis, for his timely assistance in the development of this thesis. A very special thanks goes to Lt Col William Baker. Without his guidance and help with the math, this thesis could not have been accomplished. Also, I would like to thank Dr. Robert J. Papa of Rome Air Development Center (RADC), a noted author in the field of rough surface scattering, for his valued inputs; John, Pam and the rest of the outstanding staff of the AFIT Library for their never ending support; and Capt Thomas Roseman for his can do approach to problems posed in our discussions of our theses.

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Finally, I would like to thank my wife, Alice, my daughter, Christina, and my son, Eric, for their valued support and understanding during this project.

Clarence E. Reif

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List of Symbols and Acronyms

Symbol/Acronym

E - electric field intensity

$\Gamma()$ - gamma function

h^2 - variance of the surface heights

I_n - modified bessel function of first kind, order n

J_n - bessel function of order n

j - square root of -1

jpdf - joint probability density function

K_n - modified bessel function of second kind, order n

k_o - free space wave number

ℓ - correlation length

μ - $\cot\theta$

pdf - probability density function

p - surface slope x direction

q - surface slope y direction

r - distance between two points on the surface (argument of $\rho()$)

ρ - correlation coefficient (also $\rho(r)$)

RCS - radar cross section

sgn() - signum function

ω - radian frequency of the electromagnetic field

$\langle\ \rangle$ - expected value operator

$\chi_1()$, $\chi_2()$ - marginal and joint characteristic functions

ζ_1 - surface height random variable at point (x_1, y_1)

ζ_2 - surface height random variable at point (x_2, y_2)

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Abstract

The purpose of this study was to develop a shadowing function for a rough surface model that was based upon an exponential probability density function (pdf). This study had four critical parts:

(1) Derive the characteristic function and the joint slope probability density function for a second-order exponential pdf. (2) Write a computer program to numerically compute the shadowing function for the second-order exponential pdf surface model. (3) Compare the shadowing function obtained with an 'exponential like' and Gaussian pdf. (4) Also, investigate a discrepancy in the mean-square total slope term.

The study found that the Shadowing function for the exponential pdf had a lower value than the shadowing function for the 'exponential like' and gaussian pdfs. The shadowing functions for the 'exponential like' and Gaussian pdf were found to have about the same value. This study also found that the shadowing function is sensitive to the value of the mean-square slope term used.

ELECTROMAGNETIC SHADOWING BY AN EXPONENTIALLY DISTRIBUTED ROUGH SURFACE

I. Introduction

In recent years, the performance of a complex and versatile radar system has become more sensitive to its operational environment. This has created a need for accurate mathematical models of the environment in which radar systems are used. For example, a model for electromagnetic scattering by a randomly rough surface should account for multipath effects and shadowing of the incident as well as the reflected wave. The rough surface models must, of course, be mathematically sound and preferably in agreement with measured data.

Early models used a Gaussian random probability density function (pdf) because of their analytical convenience [9:1274]. However, different surface models are needed for different magnitudes of surface roughness. For example, Gaussian random surface models do not adequately model sea-ice fields and some kinds of rough terrain because of their sharp peaks and steep slopes. To accurately model these surfaces, a joint non-Gaussian random probability density function is preferred [10:788].

In this connection, Barrick introduced the following exponential joint surface-height probability density function to model these surfaces.

$$P_E(\zeta_1,\zeta_2) = \frac{3}{2\pi h \left[1 - \rho^2(r)\right]^{1/2}}$$

$$\times \quad \exp \left\{ - \left[\frac{\zeta_1^2 - 2\zeta_1\zeta_2\rho(r) + \zeta_2^2}{\frac{1}{3}h^2\left[1 - \rho^2(r)\right]} \right]^{1/2} \right\} \tag{1}$$

Here, ζ_1 and ζ_2, which are functions of x and y, denote two heights of the rough surface at points 1 and 2 which are separated by a distance r. $\rho(r)$ is the surface-height correlation function and h^2 is the mean-square roughness height. Also, it is assumed that $\rho(r)$ does not contain a steady or periodic component [16:720,721].

Pyati stated that Eq (1) does not meet one of the statistical requirements of a pdf and is therefore, mathematically incorrect. Hence, Pyati provided the following second-order exponential pdf that does meet all the statistical conditions [14:8].

$$P_E(\zeta_1,\zeta_2;\rho) = \frac{1}{\left[4h^2\left(1-\rho^2\right)\right]} \exp\left\{-\left[\frac{|\zeta_1| + |\zeta_2|}{h\left(1-\rho^2\right)}\right]\right\}$$

$$\times \left\{ I_o\left[\frac{2\rho\left(|\zeta_1|\;|\zeta_2|\right)^{1/2}}{h\left(1-\rho^2\right)}\right] + \frac{8}{\pi^2}\,\mathrm{sgn}(\zeta_1)\;\mathrm{sgn}(\zeta_2)\right.$$

$$\left. \times \sum_{n=o}^{\infty} \frac{1}{\left(2n+1\right)^2}\, I_{2n+1}\left[\frac{2\rho\left(|\zeta_1|\;|\zeta_2|\right)^{1/2}}{h\left(1-\rho^2\right)}\right]\right\} \qquad (2)$$

Here, $\mathrm{sgn}(\zeta)$ is the signum function which is equal to plus one for positive ζ and minus one if ζ is negative. I_k is a modified Bessel function of the first kind of order k. ρ is $\rho(r)$, which is the surface-height correlation function and h^2 is the mean-square roughness height.

<u>Problem</u>

The purpose of this study is to investigate shadowing using Pyati's second-order exponential pdf. Here, shadowing is defined as, "the screening of parts of the surface by other parts interrupting the illuminating radiation" [4:384]. The results of this investigation will be compared with those obtained by Brown when he used Barrick's 'exponential like' pdf, Eq (1) [10:788-790]. It may be noted

that among others, Brown has used Barrick's Eq (1) in his investigation.

Assumptions

It will be assumed that the incident wave striking the surface is plane. This assumption is satisfied by placing the receiving and transmitting antennas sufficiently far from the rough surface. The wavelength of the incident wave will be assumed to be much shorter than the dimensions of the illuminated surface area. Also, the randomly rough surface will be considered to be a perfect conductor.

Scope

Although the problem of electromagnetic scattering and shadowing is very extensive, this investigation will only consider the study of Pyati's second-order exponential pdf following Brown's method. A comparison between the shadowing results when using Barrick's and Pyati's second-order pdfs will be made.

Shadowing Overview

Scattering coefficients of rough surfaces are prone to large errors if shadowing is neglected [4:384]. To correct for these errors, Beckmann used a shadowing function that was equal to one for illuminated areas and equal to zero for shadowed areas [4:384]. However, because of its neglect of the spectrum of shadows caused by diffraction, Beckmann's shadowing function is only valid in the high frequency limit

where the wavelength approaches zero, which is the same restriction geometrical optics has [4:388]. Beckmann concluded that fields scattered by a rough surface are strongly affected by shadowing, especially for large angles of incidence.

Expanding on Beckmann's work, Smith developed a shadowing function based upon the geometry of self shadowing. Smith's results agreed closely with a simulation of a Gaussian random surface by Brockelman and Hagfors [19:668]. Brown simplified Smith's shadowing function by assuming that the height and slopes of the surface were independent random variables. Using this assumption, Brown developed a shadowing function that can be used with a non-Gaussian probability density function [10:788].

Approach

The approach of this investigation will consists of using Brown's shadowing function

$$R(\theta) = \left[1 + \int_{\mu}^{\infty} \left(q/\mu - 1\right) P_2\left(q\right) dq \right]^{-1} \qquad (3)$$

to obtain the shadowing function for Pyati's exponential pdf. In Eq (3), μ is equal to the $\cot\theta$, where θ is the incidence angle, and $P_2(q)$ is the probability density of the surface slope q. $P_2(q)$ will be derived for Pyati's exponential pdf using a relationship established by Barrick

between the slope probability density function and the average of the integral for the average scattered power. The computed shadowing function will be compared to the results that Brown obtained for Barrick's 'exponential like' joint probability density function and also with a Gaussian probability density function.

In Chapter II, an overview of electromagnetic scattering theory and an in-depth look at Brown's shadowing function will be presented.

Chapter III contains the details of applying Brown's shadowing function to Pyati's pdf. Appendices A and B contain the details of obtaining the characteristic function of Pyati's exponential pdf. Appendix C pertains to computer programs written to numerically compute the shadowing function.

In Chapter IV, the results of applying Brown's shadowing function to Pyati's exponential pdf are presented. Also, the development of Brown's shadowing function for a Gaussian pdf is given in Appendix D.

Chapter V contains conclusions based upon the theoretical analysis and computational results. Recommendations for future research are also discussed.

II. Theory

This chapter contains the development of Brown's backscattering shadowing function. To develop his shadowing function, Brown used Smith's geometrical shadowing approach and also, assumed that the surface heights and slopes were statistically independent [10:788]. This chapter will discuss the theory of electromagnetic scattering first, followed by the development of Brown's shadowing function.

Electromagnetic Scattering

One analytical approach that has been applied to the study of the scattering of electromagnetic waves by a random rough surface is the Kirchhoff approximation approach. Kirchhoff approximation, also *known* as *physical* optics, approximates the fields at any point on the surface by the fields that would be present on the tangent plane at that point. This approximation is why a large radius of curvature relative to the incident wavelength at every point on the surface is required.

For the case of backscattering, when the physical optics approximation is used in the two dimensional case and the surface is assumed to be a perfect conductor, the electric field at a point $\mathcal{P}$ in the far field at a distance R from the surface is given by

$$E(\mathcal{P}) = \frac{jkE^{i}\, e^{jkR}}{2\pi R} \sec(\theta) \int_{S} \exp\left[(2j)\left(k_x x + k_z \zeta\right)\right] dxdy \qquad (4)$$

where $k_z = k\cos(\theta)$, $k_x = k\sin(\theta)$ with k being the free-space wave number. ζ is the random surface height variable and j is the square root of minus one [15:38].

The average Radar Cross Section (RCS) of the surface that produced the electromagnetic field given by Eq (4) is given as

$$\langle RCS \rangle = 2k^2 \sec^2(\theta) \int_0^{\infty} rJ_0\left(r\sqrt{v_x}\right)\left[\chi_2 - \chi_1^2\right] dr \qquad (5)$$

where χ_1 and χ_2 are the joint and marginal characteristic functions of the surface pdf and v_x is equal to $2k\sin(\theta)$. J_0 is a Bessel function of the first kind of order zero.

A shadowing function, $R(\theta)$, is used to correct the RCS of the surface for errors caused by shadowing of the electromagnetic waves by parts of the rough surface. The RCS of a two dimensional surface that is corrected for shadowing is given as

$$\langle RCS \rangle = 2k^2\ R(\theta)\ \sec^2(\theta) \int_0^{\infty} rJ_0\left(r\sqrt{v_x}\right)\left[\chi_2 - \chi_1^2\right] dr \qquad (6)$$

A more detail explanation of electromagnetic scattering theory can be found in Roseman [15]. Roseman computes the RCS of a surface described by Pyati's exponential pdf and then compares the RCS obtain with the RCS of a surface described by Barrick's 'exponential like' pdf.

Brown's Shadowing Function

Smith derived a shadowing function defined as the probability that a point on a randomly rough surface will not be shadowed by other parts of the surface. The geometry of the problem is shown in Figure 1. The point *F*, located on the surface, is a distance of ξ_0 above the mean plane of

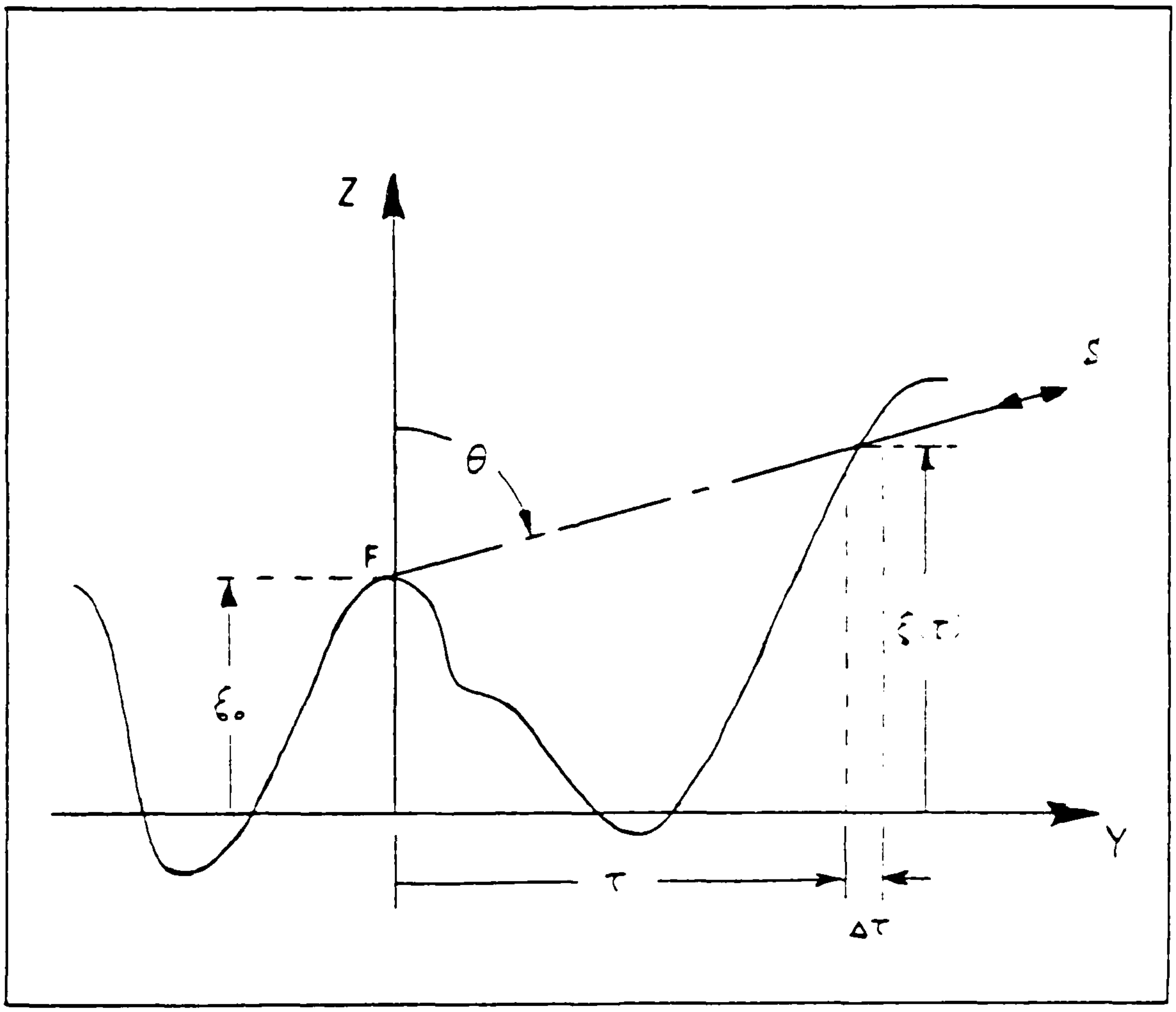

Figure 1. Shadowing Geometry of the Rough Surface. *SF* is the Incident Ray. [7:23]

the surface. The local slopes at the point F are p_0 in the x, z plane and q_0 in the y, z plane. The origin of the x, y, and z coordinates is located on the mean plane of the surface directly below the point F. Also, the incident electromagnetic field is orientated in the $x = 0$ plane. Since only the surface to the right of F can shadow F, Smith's shadowing function $S(F,\theta)$ is equivalent to, "the probability that no part of the surface to the right of F will intersect the ray FS" and is given by [19:669]

$$S(F,\theta) = h(\mu - q_0)\exp\left\{ -\int_0^{\tau} g(\tau)d\tau \right\} \tag{7}$$

In Eq. (7), h() is the unit step function, $\mu = \cot\theta$ where θ is the incident angle measured from vertical, and q_0 is the slope of the surface in the y direction at F. Also, $g(\tau)\Delta\tau$ is, "the conditional probability that the surface in $\Delta\tau$ *will* intersect the ray FS given that it does not in the interval τ" and the result is [19:669]

$$g(\tau)\Delta\tau = \frac{\Delta\tau \int_{\mu}^{\infty} dq\ (q-\mu)\left[P_3(\xi, q|F,\tau)\right]_{\xi=\xi_0+\mu\tau}}{\int_{-\infty}^{+\infty} dq \int_{-\infty}^{\xi_0+\mu\tau} d\xi\ P_3(\xi, q|F,\tau)} \tag{8}$$

Brown and Smith evaluated Eq (8) by assuming that the heights and slopes at F, ($y = 0$), were uncorrelated with

those at $y = \tau$. Brown also assumed that the decorrelation implied that the heights and slopes were independent random variables [10:788-789]. This assumption is valid for Gaussian distributions; however, it is not necessarily valid for all other surface distributions [13:136]. However, this assumption allowed P_3 to be written as [10:789]

$$P_3(\xi, q \,|\, F, \tau) = P_1(\xi) \; P_2(q) \tag{9}$$

Here, $P_1(\xi)$ is the height probability density function, and $P_2(q)$ is the y slope probability density function. $P_2(q)$ is derived from $P_{22}(p,q)$, the joint pdf of the x and y slopes, and is given by [10:789]

$$P_2(q) = \int_{-\infty}^{+\infty} P_{22}(p,q) \; dp \tag{10}$$

Using Eqs (8) through (10), Brown developed a shadowing function, $R(\theta)$, that is defined as the, "probability that a backscattering specular point on the surface will not be shadowed" [10:788]. This function is given by

$$R(\theta) = \left\{ 1 + \int_{\mu}^{\infty} \left[q/\mu - 1 \right] \; P_2(q) \; dq \right\}^{-1} \tag{11}$$

In Eq (11), $\mu = \cot\theta$, where θ is the incidence angle, and $P_2(q)$ is the probability density of the surface slope.

This equation will be used in the next chapter to compute the shadowing function for Pyati's exponential pdf, Barrick's 'exponential like' pdf and a Gaussian pdf. Shadowing results will be presented and compared for the three pdfs in Chapter IV.

III. Analysis

Objective

In this chapter, the shadowing function for Pyati's exponential bivariate pdf, Eq (2), will be computed and compared to Barrick's 'exponential like' pdf and a Gaussian pdf.

In order to determine the shadowing function for Pyati's pdf, the joint probability density function of the x and y slopes, $P_{22}(p,q)$, must first be computed for Pyati's pdf. Once P_{22} is known, it will be used to compute the shadowing function.

Joint Slope Probability Density Function

Barrick established the following relationship, valid in the high frequency limit, between the slope probability density function and the average of the integral for the average scattered power of a rough surface.

$$\lim_{k_s \to \infty} \left\{ \frac{k_s^2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[ik_s \xi_x \left(x_1 - x_2\right) + ik_s \xi_y \left(y_1 - y_2\right)\right] \right.$$

$$\left\langle \exp ik_s \xi_z \left[\zeta\left(x_1, y_1\right) - \zeta\left(x_2, y_2\right)\right]\right\rangle$$

$$\left. d\left(x_1 - x_2\right) d\left(y_1 - y_2\right) \right\} = \frac{4\pi}{\xi_z^2} P\left(\frac{\xi_x}{\xi_z}, \frac{\xi_y}{\xi_z}\right) \qquad (12)$$

In Eq (12), $\zeta(x_1,y_1)$ and $\zeta(x_2,y_2)$ are the surface heights at points 1 and 2 on the surface. In Barrick's notation $\xi_x = \sin(\theta_i) - \sin(\theta_s)\cos(\emptyset_s)$, $\xi_y = -\sin(\theta_s)\sin(\emptyset_s)$ and $\xi_z = -\cos(\theta_i) - \cos(\theta_s)$, where θ_i and θ_s are the incidence and scattered angles respectively. The angular brackets $\langle\,\rangle$ denotes an ensemble average. The term k_o is the free-space wave number $2\pi/\lambda$. Also, the time dependence $\exp(-j\omega t)$ has been assumed and omitted [2:1728].

The function P on the right side of Eq (12) is the joint probability density function for the surface slopes which is needed to compute Brown's shadowing function. To evaluate the left side of Eq (12), the separation distance between the points on the surface (x_1,y_1) and (x_2,y_2) are converted to polar coordinates. If r is the separation distance and φ is the direction angle from (x_1,y_1) to (x_2,y_2) then

$$r = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{1/2}$$

$$x_1 - x_2 = r\cos\varphi$$

$$y_1 - y_2 = r\sin\varphi \qquad (13)$$

Substituting Eq (13) in the left side of Eq (12) and omitting the limit, with an understanding that it still applies, results in

$$\frac{k_\circ^2}{\pi} \int_0^{2\pi} \int_0^{\infty} \exp\left[ik_\circ \xi_x r \cos\varphi + ik_\circ \xi_y r \sin\varphi\right]$$

$$\times \quad \left\langle \exp\ ik_\circ \xi_z \left[\zeta\left(x_1, y_1\right) - \zeta\left(x_2, y_2\right)\right]\right\rangle\ r\ dr\ d\varphi \qquad (14)$$

Integrating Eq (14) over φ yields

$$2k_\circ^2 \int_0^{\infty} J_0\left[r \sqrt{\left(v_x^2 + v_y^2\right)}\ \right]$$

$$\times \quad \left\langle \exp\ iv_z \left[\zeta\left(x_1, y_1\right) - \zeta\left(x_2, y_2\right)\right]\right\rangle\ r\ dr \qquad (15)$$

where $v_x = k_\circ \xi_x$, $v_y = k_\circ \xi_y$, and $v_z = k_\circ \xi_z$. Also, J_0 is a zero order Bessel function.

Barrick showed that the surface slopes in the x and y direction can be written as $-\xi_x/\xi_z$ and $-\xi_y/\xi_z$ respectively [16:720]. Therefore, Brown's p and q slope terms are equal to $-v_x/v_z$ and $-v_y/v_z$. Combining this with Eq (12) and (15), the joint probability density for the surface slopes can be obtained.

$$P_{22}(p, q) = \frac{v_z^2}{2\pi} \int_0^{\infty} J_0\left[r\ v_z \sqrt{\left(\ p^2 + q^2\right)}\ \right]$$

$$\times \quad \left\langle \exp\ iv_z \left[\zeta\left(x_1, y_1\right) - \zeta\left(x_2, y_2\right)\right]\right\rangle\ r\ dr \qquad (16)$$

It is noted that the term in the angular brackets is equal to $\chi_2(v_z, -v_z)$. Here, χ_2 is the characteristic function of the joint pdfs [5:78-79]. Eq (16) becomes

$$P_{22}(p,q) = \frac{v_z^2}{2\pi} \int_0^\infty J_0\left[r\, v_z \sqrt{(p^2 + q^2)}\right]$$

$$\times \quad \left[\chi_2(v_z, -v_z; r)\right] r\, dr \qquad (17)$$

Equation (17) is the joint probability density function of the surface slopes that was used to compute the shadowing function for Pyati's exponential pdf. The next section of this chapter expands on the characteristic function.

Characteristic Function

The second order characteristic function of the random variables ζ_1 and ζ_2 is defined as

$$\chi_2(v_1, v_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_2(\zeta_1, \zeta_2) \exp\left(i v_1 \zeta_1 + i v_2 \zeta_2 \right) d\zeta_1 \, d\zeta_2 \qquad (18)$$

where $P_2(\zeta_1, \zeta_2)$ is the joint pdf of ζ_1 and ζ_2 [5:189]. Also, $\zeta_1 = \zeta(x_1, y_1)$, $\zeta_2 = \zeta(x_2, y_2)$ and $v = v_z$ are used.

The joint pdfs are a functions of the surface-height correlation coefficient $\rho(r)$. Therefore, the characteristic functions are also a function of the correlation coefficient. The variable r is the separation distance between the two points, ζ_1 and ζ_2, on the randomly rough surface. As r approaches infinity, the interdependence of ζ_1 and ζ_2

decreases. At infinity, the terms ζ_1 and ζ_2 are independent [3:129].

Pyati's Exponential Probability Density Function

Pyati's exponential pdf is [14]

$$P_E(\zeta_1,\zeta_2;\rho) = \frac{1}{\left[4h^2\left(1-\rho^2\right)\right]} \exp\left\{ -\left[\frac{|\zeta_1| + |\zeta_2|}{h\left(1-\rho^2\right)} \right] \right\}$$

$$\times \left\{ I_o\left[\frac{2\rho\left(|\zeta_1|\ |\zeta_2| \right)^{1/2}}{h\left(1-\rho^2\right)}\right] + \frac{8}{\pi^2} \operatorname{sgn}(\zeta_1)\ \operatorname{sgn}(\zeta_2)\right.$$

$$\left. \times \sum_{n=o}^{\infty} \frac{1}{\left(2n+1\right)^2}\ I_{2n+1}\left[\frac{2\rho\left(|\zeta_1|\ |\zeta_2| \right)^{1/2}}{h\left(1-\rho^2\right)}\right] \right\} \qquad (19)$$

where I_k is a modified Bessel function of the first kind of order k, h^2 is the mean-square roughness and also, $\operatorname{sgn}(\zeta)$ is equal to plus one for ζ greater than zero and equal to minus one for ζ less than zero. The formidable task of solving for the characteristic function is shown in Appendices A and B and the final result is

$$\gamma_2(v,-v) = \frac{1}{2}\left\{ \frac{1 + (1-\rho^2)h^2v^2}{(1-\rho^2)^2h^4v^4 + 2h^2v^2 - 2\rho^2h^2v^2 + 1} \right.$$

$$+ \frac{1 - (1-\rho^2)h^2v^2}{(1-\rho^2)^2h^4v^4 + 2h^2v^2 + 2\rho^2h^2v^2 + 1}$$

$$+ \frac{8}{\pi^2} (1-\rho^2) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\rho^{2n+1+2k}}{(2n+1)^2\ k!\ (2n+1+k)!}$$

$$\times \frac{\Gamma^2(n+k+1.5)\ \sin^2\left[(n+k+1.5)\theta\right]}{\left[1 + ((1-\rho^2)hv)^2\right]^{(n+k+1.5)}} \tag{20}$$

$$\theta = \tan^{-1}\left[(1-\rho^2)hv\right] = \cos^{-1}\left[\frac{1}{\left[1 + ((1-\rho^2)hv)^2\right]^{1/2}}\right]$$

$$= \sin^{-1}\left[\frac{(1-\rho^2)hv}{\left[1 + ((1-\rho^2)hv)^2\right]^{1/2}}\right] \tag{21}$$

The joint probability density function for the surface slopes for Pyati's pdf was obtained by substituting Eqs (20) and (21) into Eq (17). However, due to the complexity of this function, it could not be integrated in a closed form. Therefore the function was evaluated numerically on a computer, details of which may be found in Appendix C.

To gain confidence in the accuracy of the numerical integrations, the algorithm was applied to the Gaussian pdf

for which closed form solutions are known. Details of the closed form solution for the Gaussian pdf are given in Appendix D. The difference between the numerical solution and the closed form solution was less than one tenth of a percent. Therefore, the algorithm was consider sufficient to do the integrations.

Correlation Coefficient

The correlation coefficient, $\rho(r)$, also called the autocorrelation, gives the correlation between ζ_1 and ζ_2. One form of the correlation coefficient is the Gaussian given by $\rho(r) = \exp\left[-r^2/\ell^2\right]$, where ℓ is the correlation length. Because of its rapid convergence and its zero slope at the origin, the Gaussian correlation coefficient has found wide use [1:103].

Mean-Square Total Slope

Beckmann, Smith, Kong and others used the following equation for the mean-square total slope of the rough surface in their work [5:193], [11:540], [18:669]

$$w^2 = -h^2 \left. \frac{\partial^2 \rho(r)}{\partial r^2} \right|_{r=0} \qquad (22)$$

and for the Gaussian case $w^2 = 2h^2/\ell^2$. However, Barrick stated that since $r^2 = (x_1-x_2)^2 + (y_1-y_2)^2$, the mean-square total slope should be taken as [1:104]

$$w^2 = -h^2 \left[\left. \frac{\partial \rho \left\{ \left(r_x^2 + r_y^2 \right)^{1/2} \right\}}{\partial \rho_x^2} \right|_{\substack{r_x=0 \\ r_y=0}} + \left. \frac{\partial \rho \left\{ \left(r_x^2 + r_y^2 \right)^{1/2} \right\}}{\partial \rho_y^2} \right|_{\substack{r_x=0 \\ r_y=0}} \right] \qquad (23)$$

and hence, $w^2 = 4h^2/\ell^2$ when $\rho(r)$ is Gaussian. This indicates a factor of two variation between Barrick's and Beckmann's term for the mean-square total slope of the rough surface. This difference will be discussed in the next chapter.

IV. Results

Introduction

In this chapter, following the method used by Brown, the shadowing function for Pyati's exponential pdf and Barrick's 'exponential like' pdf are compared. The first section of this chapter examines the impact of the factor of two variation between Barrick's and Beckmann's mean-square slope. This difference between Barrick's and Beckmann's mean-square slope term was discussed in the last chapter. Next, the normalized marginal slope density, $P_2(q)$, is graphed for Pyati's, Barrick's and a Gaussian pdf. Finally, the shadowing functions are graphed for all three pdfs for several values of the mean-square total slope term.

Factor of Two Difference

To avoid confusion, Barrick's mean-square surface slope term will be denoted by w^2 and Beckmann's by m^2. Therefore, the following equation can be written where h^2 is the mean-square roughness height and ℓ is the surface correlation length.

$$w^2 = 2m^2 = \frac{4h^2}{\ell^2} \tag{24}$$

Using this notation, Beckmann's joint density function of the surface slopes for the Gaussian surface is [19:663]

$$P_{22}(p,q) = \frac{1}{2\pi m^2} \exp\left[-\left(\frac{p^2+q^2}{2m^2}\right)\right] \tag{25}$$

and that of Barrick's is [16:720-721]

$$P_{22}(p,q) = \frac{1}{\pi w^2} \exp\left[-\left(\frac{p^2+q^2}{w^2}\right)\right] \tag{26}$$

for $-\infty < p,q < \infty$.

The marginal density, $P_2(q)$, is obtained by integrating $P_{22}(p,q)$ over all values of p. The slope marginal pdf for Beckmann's and Barrick's Gaussian pdf are given in Eqs (27) and (28) respectively.

$$P_2(q) = \frac{1}{m\sqrt{2\pi}} \exp\left[-\left(\frac{q^2}{2m^2}\right)\right] \tag{27}$$

$$P_2(q) = \frac{1}{w\sqrt{\pi}} \exp\left[-\left(\frac{q^2}{w^2}\right)\right] \tag{28}$$

For completeness, the joint surface slope and the surface slope marginal for Barrick's 'exponential like' pdf are presented in Eqs (29) and (30) [10:789].

$$P_{22}(p,q) = \frac{3}{\pi w^2} \exp\left[-\sqrt{6}\left(\frac{p^2+q^2}{w^2}\right)^{1/2}\right] \tag{29}$$

$$P_2(q) = \frac{6}{\pi w^2} \; |q| \; K_1\left[\frac{\sqrt{6} \; |q|}{w} \right] \tag{30}$$

$K_1()$ is a modified Bessel function of the second kind of order one. The $P_2(q)$ function for Pyati's pdf was computed numerically from equations (17) and (20).

To demonstrate the effect the factor of two has on the results, both Beckmann's and Barrick's Gaussian marginal slope densities are plotted in Figure 2. The normalized densities of Eqs (27) and (28) are plotted as a function of $P_2(q)$ times w against the normalized slope q/w for Barrick's and as a function of $P_2(q)$ times m against the normalized slope q/m for Beckmann's. Brown used this method to compare the slope probabilities of Eqs (27) and (30) [3:789].

It is readily apparent from Figure 2 that the factor of two makes a difference in the slope probabilities. Comparing Barrick's Gaussian pdf to Beckmann's, Barrick's pdf has a greater probability that a smaller q will occur and a decreased probability for a larger q. Small q denotes a small slope value.

The difference between the two pdf in Figure 2 is great enough to re-examine Brown's results. Therefore, the next sections of this chapter will compare Barrick's Gaussian and 'exponential like' pdfs. Also, Pyati's exponential pdf will be included in the comparison.

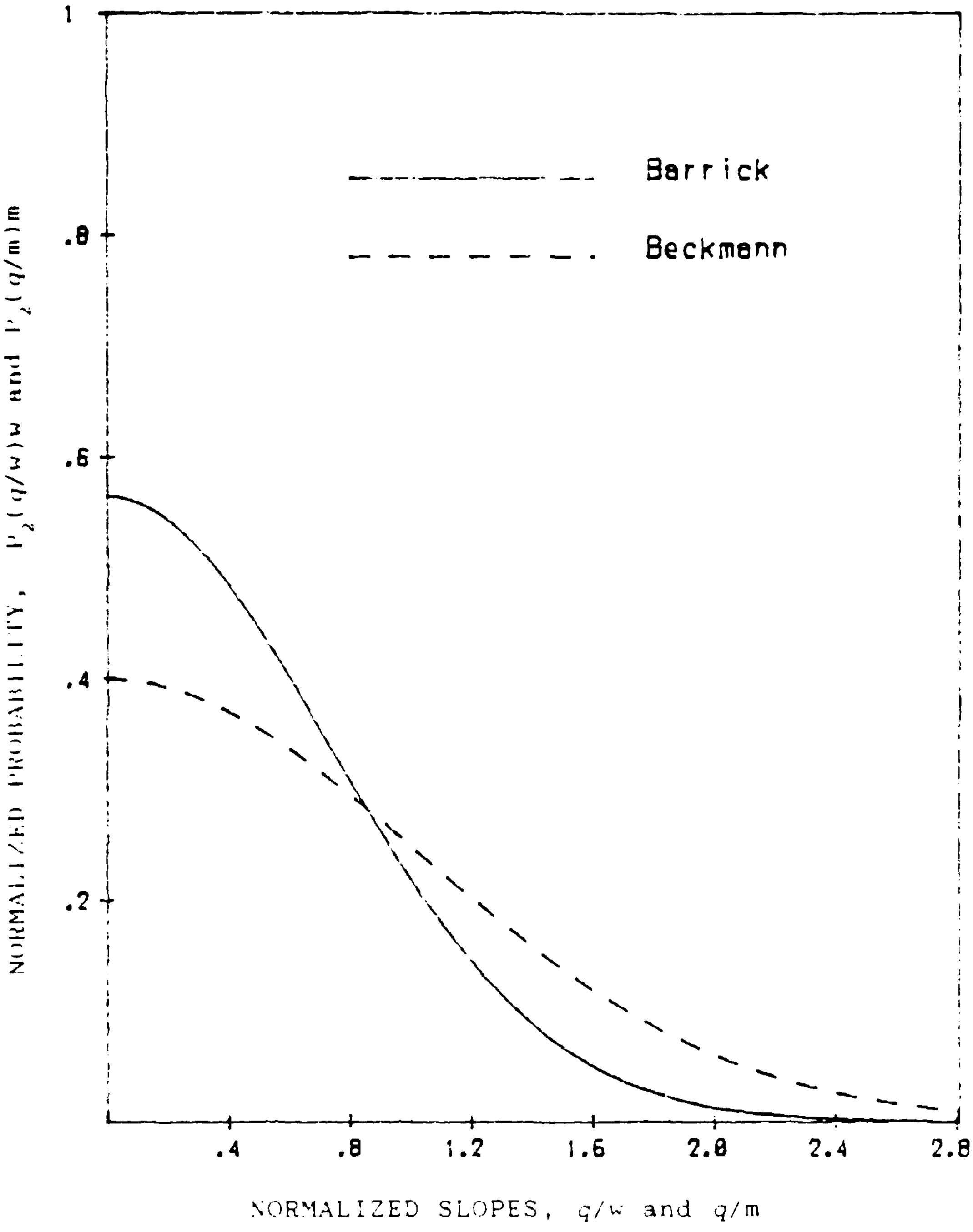

Figure 2. Comparison of Barrick's and Beckmann's Gaussian Marginal Slope Density

Slope Probability Density Function

For backscattering to occur, the surface must be perpendicular to the incident signal. Therefore, the slope q must equal minus the tangent of the incident angle theta. But, since $P_2(q)$ is an even function, q can be written as the tangent of the incident angle when computing the probability of the surface slope. Consequently, when theta is equal to zero, the surface slope is horizontal.

The normalized densities of Eqs (28), (30) and the numerical solution of Pyati's pdf are plotted in Figure 3 as a function of $P_2(q)$ times w against the normalized slope q/w. Figure 3 shows that all three pdfs have their largest probability values for small values of q/w and that their probability values approach zero as q/w increases. Small values of q/w corresponds to small slope angles. Pyati's and the Gaussian pdf have about the same probability for small values of q/w, while Barrick's 'exponential like' has a much greater probability.

Brown states that there is an intuitive reason why Barrick's 'exponential like' pdf has a much greater probability for small slopes. According to Brown, having the higher probability for small slopes is, "in agreement with one intuitive approach for generating a surface characterized by Eq (29), e.g., one strongly filters all surface height excursions below a certain level to eliminate the possibility of large negative height excursions. This

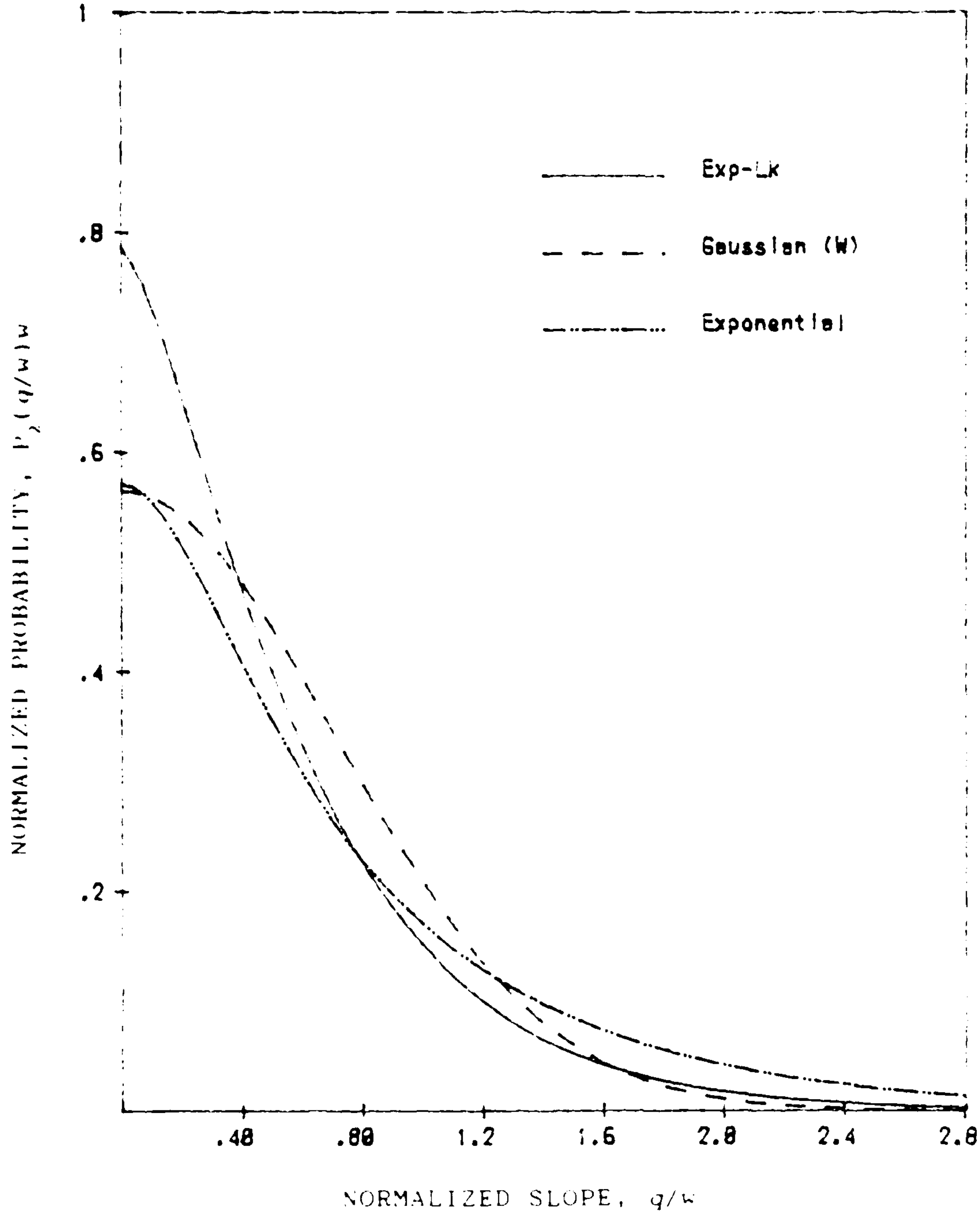

Figure 3. Normalized Marginal Slope Densities for Barrick's 'Exponential Like', Pyati's Exponential and a Gaussian PDF

process increases the probability of small slopes at the expense of the larger slopes" [10:789].

The probability for large slopes should be greater for an exponential slope pdf than for a Gaussian slope pdf. Looking at Figure 3, Pyati's exponential pdf has a slight but significantly higher probability of larger slopes than does Barrick's Gaussian and 'exponential like' pdfs. However, the cumulative probability distribution, area under each graph, is the same and equal to a half for each pdf. The area under the curves would be one if they were graphed from minus infinity to plus infinity.

Figure 4 is similar to Figure 3 except that the mean-square slope w has been set equal to one and the q slope term converted to its slope angle using the fact that the tangent of theta equals q. Therefore, Figure 4 is a graph of the probability of a slope angle verses the slope angle, given that w is equal to one. The slight but significantly higher probability of larger slope angles for Pyati's exponential pdf can be seen in Figure 4. The probability of a slope of 70 degrees with w = 1 was computed to be .0155, for Pyati's exponential, .0036 for Barrick's 'exponential like', and .0004 for the Gaussian pdf. The next section illustrates Brown's shadowing results for the three pdfs.

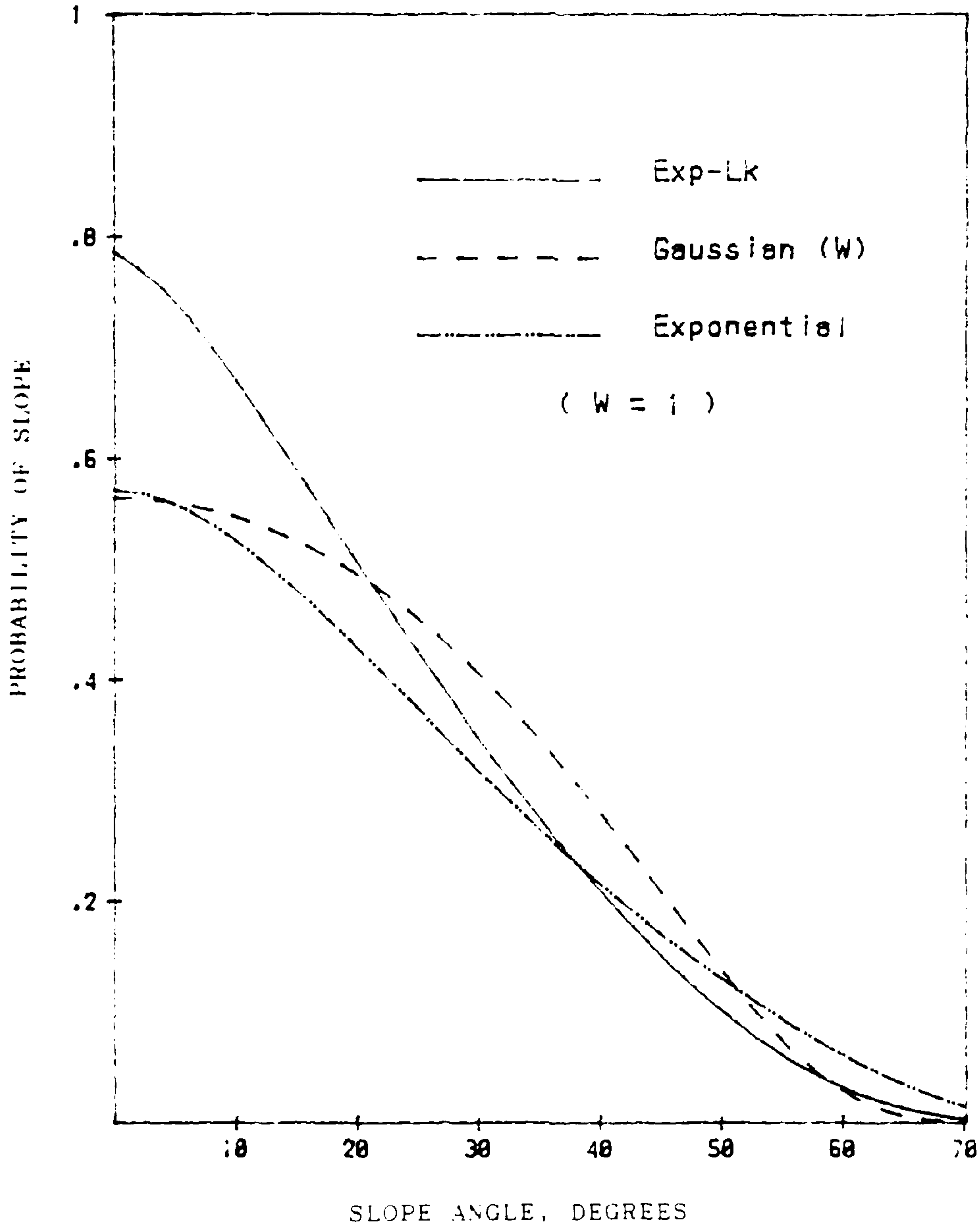

Figure 4. Slope Angle versus Probability for Barrick's 'Exponential Like', Pyati's Exponential and a Gaussian PDF with the Mean-Square Slope, W, as 1

Shadowing Function

This section contains the results of applying Brown's shadowing function

$$R(\theta) = \left\{ 1 + \int_{\mu}^{\infty} \left[q/\mu - 1 \right] P_2(q)\, dq \right\}^{-1} \tag{31}$$

to various pdfs. In Eq (31), $\mu = \cot\theta$ and $P_2(q)$ is the probability density of the surface slope. Brown defined $R(\theta)$ as the "probability that a backscattering specular point on the surface will not be shadowed" [10:788].

First, the results of the shadowing function will be compared when equations (27) and (28) are used for $P_2(q)$ in Eq (31). This will establish what effect the factor of two has on the shadowing function. Then, the results of Brown's shadowing function are compared for the three different pdfs.

When Eq (28) is used for $P_2(q)$, Eq (31) becomes

$$R(\theta) = \left\{ 1 + \frac{w}{2\mu\sqrt{\pi}} \exp\left[\frac{-\mu}{w^2} \right] - \frac{1}{2}\, \mathrm{erfc}\left[\frac{\mu}{w} \right] \right\}^{-1} \tag{32}$$

Further more, the backscattering shadowing function for Beckmann's Gaussian pdf, Eq (27), is given by

$$R(\theta) = \left\{ 1 + \frac{m}{\mu\sqrt{2\pi}} \exp\left[\frac{-\mu}{2m^2} \right] - \frac{1}{2}\, \mathrm{erfc}\left[\frac{\mu}{\sqrt{2}\, m} \right] \right\}^{-1} \tag{33}$$

Appendix D gives the details of the derivation of Eqs (32) and (33).

In Eqs (32) and (33), $\mu = \cot\theta$, where θ is the incident angle measured from the vertical axis. The variables m and w are related by $2m^2 = w^2 = 4h^2/\ell^2$, where ℓ is the surface correlation length and h^2 is the mean-square roughness height. Also, erfc($\cdot$) is the complementary error function.

The equations $2m^2 = w^2 = 4h^2/\ell^2$, can be reduced to $h^2(m) = 2h^2(w)$. Therefore, the mean-square roughness height used by Beckmann is twice the value used by Barrick when the correlation length is held constant. This greater height implies a greater probability of larger slopes for Beckmann's pdf. The probability of the larger slopes was seen in Figure 2.

If the value of the function $P_2(q)$ is increase in an interval in Eq (31), then the probability is decreased that a backscattering specular point on the surface will not be shadowed. Therefore, the shadowing function should be less for Beckmann's pdf because of the higher probability of larger angles.

As expected, Figure 5 shows that the shadowing function for Beckmann's Gaussian pdf is less then Barrick's Gaussian pdf. The shadowing functions for Beckmann and Barrick's Gaussian pdfs, plotted in Figure 5 with $w = m = 1.41$,

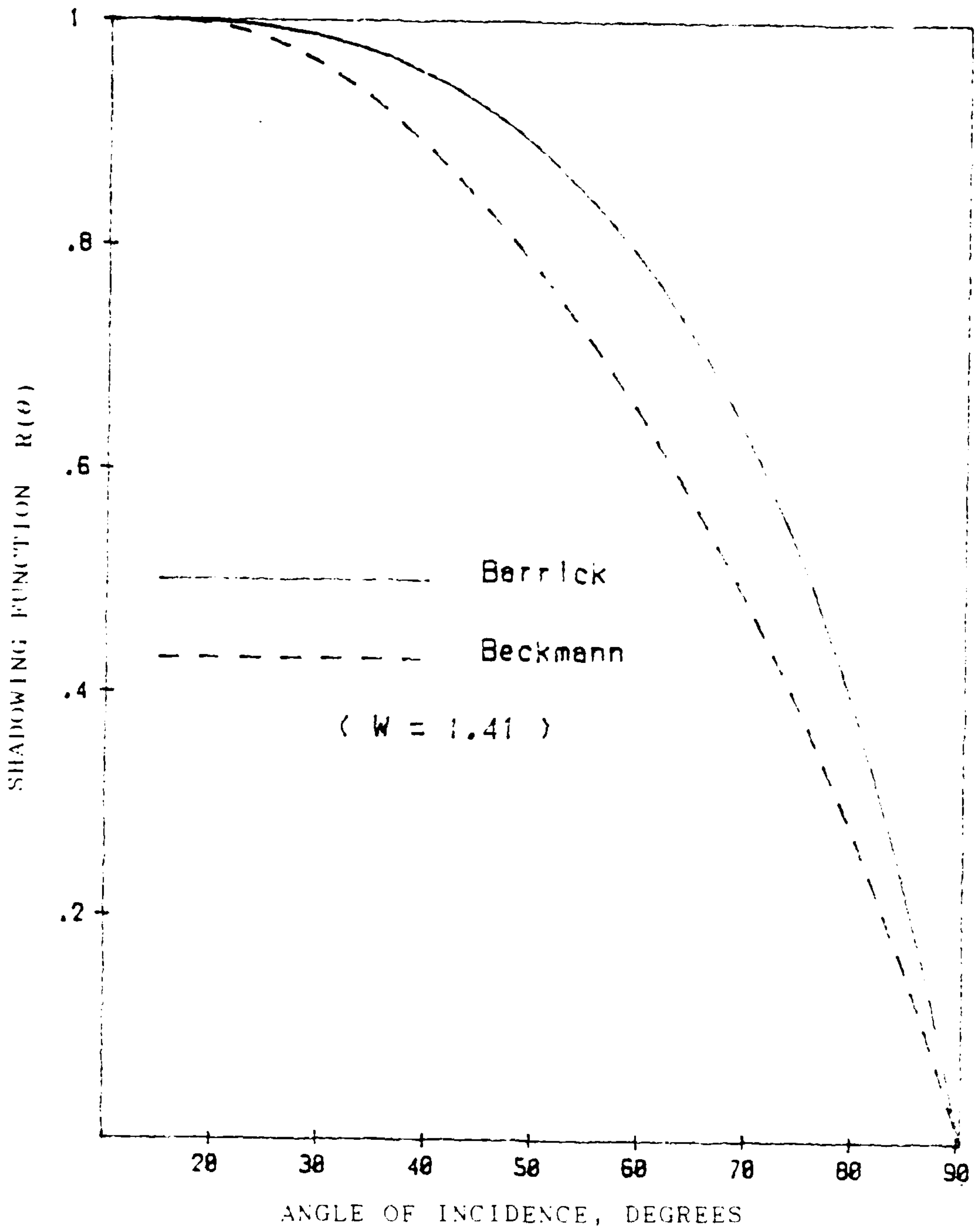

Figure 5. Backscattering Shadowing Function for Barrick's and Beckmann's Gaussian PDF with W = 1.41

shows that the factor of two greatly influences Brown's shadowing function.

Brown's shadowing function is graphed in Figure 6 for Barrick's Gaussian and 'exponential like' pdfs and Pyati's exponential pdf with a value of w equal to .71. The shadowing function for Barrick's pdfs, whose graphs actually cross each other, have about the same value. Where as, Pyati's pdf produces a shadowing function of smaller magnitude for the same value of w. The smaller shadowing function results for Pyati's exponential pdf are due to the pdf's higher probability of having larger slopes than Barrick's Gaussian and 'exponential like' pdfs.

The closed form solution for the backscattering shadowing function of Barrick's 'exponential like' pdf was obtain by Brown as [10:790]

$$R(\theta) = \left\{ \frac{x}{\pi} K_2(x) + \frac{1}{2} + \frac{x}{2} \left[K_1(x)\, L_0(x) + L_1(x)\, K_0(x) \right] \right\}^{-1} \quad (34)$$

The K_n term is the modified Bessel of the second kind of order n. The L_n symbolizes the modified Struve functions and the variable x is given by $x = \left(\sqrt{6}/w\right) \cot\theta$. Eq (34) was used to graph Barrick's 'exponential like' pdf in Figure 6.

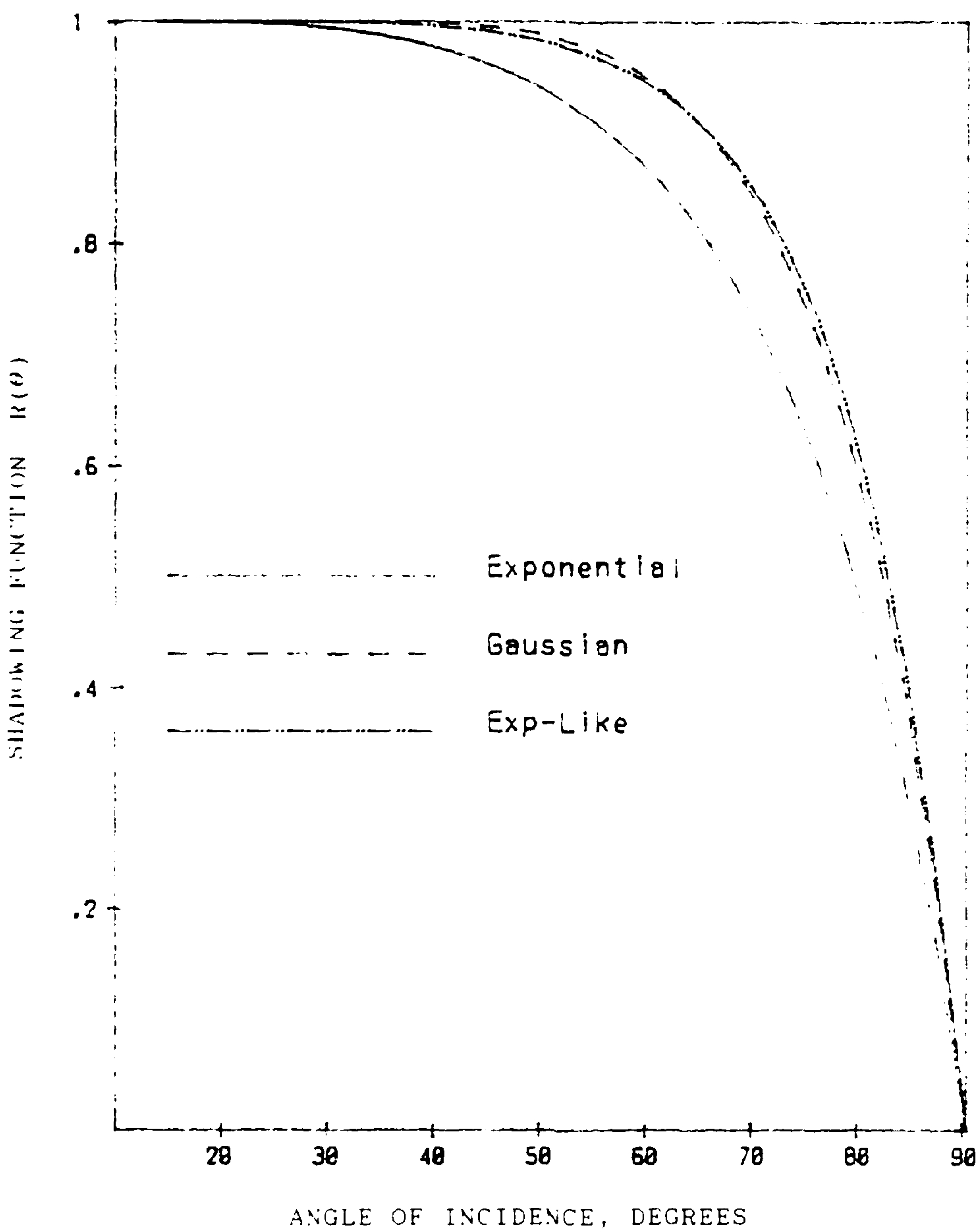

Figure 6. Backscattering Shadowing Function for Barrick's 'Exponential Like', Pyati's Exponential and a Gaussian PDF

Figure 7 is a graph of the shadowing function for Pyati's exponential probability density function for several values of the mean-square slope. Figure 7 shows that the larger the value of the mean-square slope w^2, the lower the value of the shadowing function. Figure 7 is used in conjunction with equation 6 to obtain the shadow corrected average radar cross section of the surface.

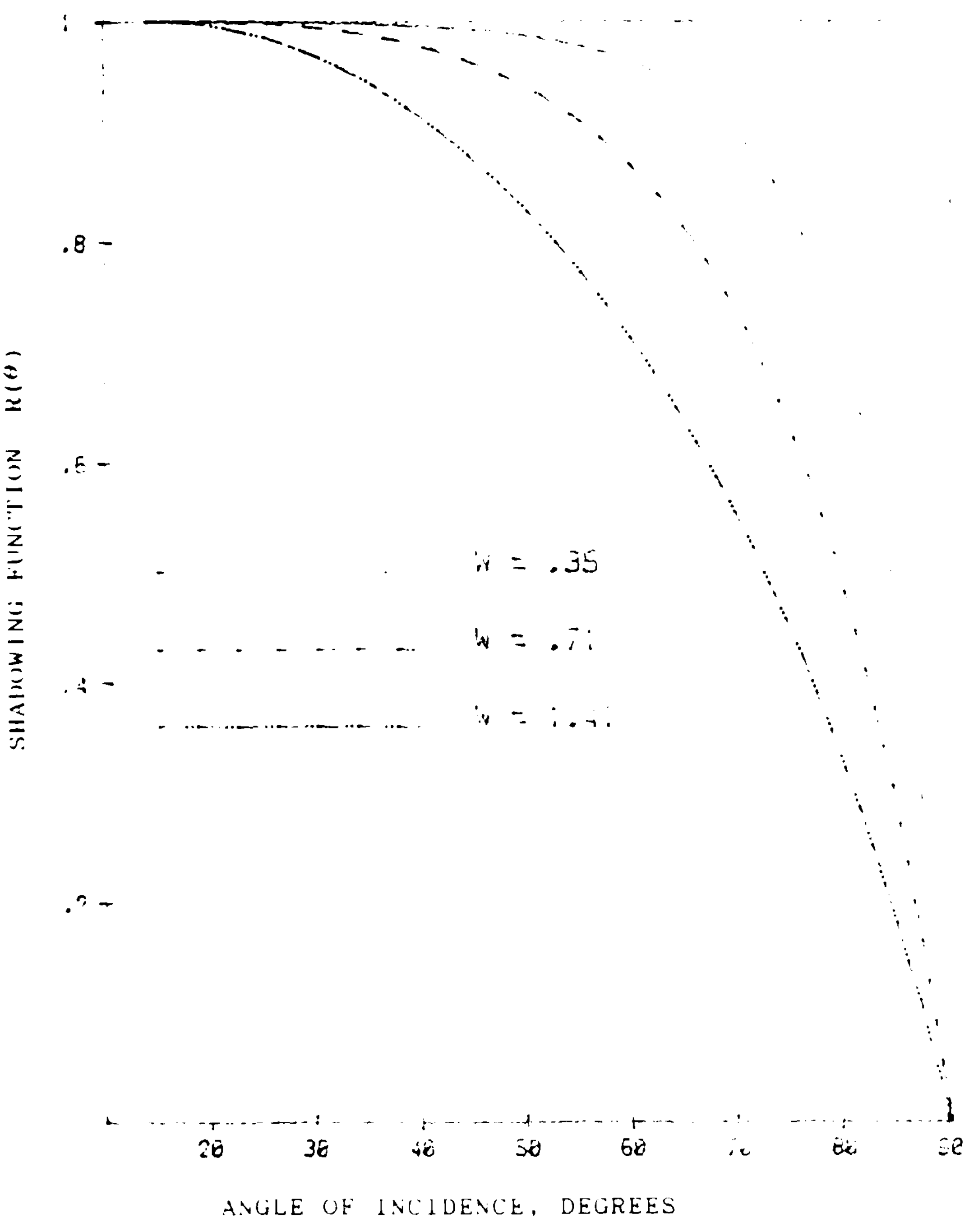

Figure 7. Shadowing Function for Pyati's Exponential Probability Density Function for Several Values of the Mean-Square Slope

V. Conclusion and Recommendations

Conclusions

The objective of this study has been to determine the shadowing function for the electromagnetic backscattering from a randomly rough surface characterized by Pyati's exponential pdf.

Based on a procedure developed by Brown, a shadowing function for Pyati's pdf was computed. The shadowing function was found to have a lower value than the shadowing function for Barrick's 'exponential like' and Gaussian pdf. It was determined that the shadowing function's lower value was due to the high probability of larger slopes for Pyati's pdf. It was also determined that the shadowing function for Barrick's 'exponential like' and Gaussian pdf have about the same values.

Recommendations

To further the discipline of electromagnetic scattering and shadowing, the following recommendations are made for future study:

1. The factor of two difference, between Barrick's and Beckmann's mean-square slope term, needs to be reviewed to determine which author is correct. This thesis showed that there is a discernible difference in the shadowing results when the different mean-square terms are used.

2. Further electromagnetic scattering and shadowing research could be pursued by building physical surface models based upon the three pdfs discussed and measuring their radar cross section using a laser set up at different angles for both the momostatic and bistatic case.
3. In this thesis much of the computation was done numerically on a computer because of the inability to derive a closed form solution of the characteristic function for Pyati's pdf. Further attempts should be made for a closed form solution.

Appendix A

Characteristic Function for the Bivariate Exponential PDF

The purpose of this appendix is to show how the characteristic function $\chi_2^E(\zeta_1,\zeta_2;\rho)$ for Pyati's two-dimensional exponential pdf Eq (2) was derived. The characteristic function $\chi_2(v_1,v_2)$ of a two-dimensional pdf $P_2(\zeta_1,\zeta_2)$ is defined as [5:189]

$$\chi_2(v_1,v_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_2(\zeta_1,\zeta_2) \exp\left[jv_1\zeta_1 + jv_2\zeta_2\right] d\zeta_1 d\zeta_2 \qquad \text{(A-1)}$$

To find the characteristic function of Pyati's two-dimensional exponential pdf, replace $P_2(\zeta_1,\zeta_2)$ in Eq (A-1) with Eq (2) and solve. The integrals that need to be solved to obtain the characteristic function are shown in Eq (A-2).

In Eq (A-2), ζ_1 and ζ_2 symbolize two random surface heights which are functions of x and y. ζ_1 and ζ_2 are separated by a distance r and where $\rho = \rho(r)$ is the surface-height correlation function. I_k is a modified Bessel function of the first kind of order k. Also, $\mathrm{sgn}(\zeta)$ is the signum function of ζ.

$$\chi_2^E(v_1, v_2; \rho) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{4h^2(1-\rho^2)} \exp\left\{-\left[\frac{|\zeta_1| + |\zeta_2|}{h(1-\rho^2)}\right]\right\}$$

$$\cdot \left\{ I_o\left[\frac{2\rho\left(|\zeta_1|\ |\zeta_2|\right)^{1/2}}{h(1-\rho^2)}\right] + \frac{8}{\pi^2}\,\mathrm{sgn}(\zeta_1)\,\mathrm{sgn}(\zeta_2)\right.$$

$$\left. \cdot \sum_{n=o}^{\infty} \frac{1}{(2n+1)^2}\, I_{2n+1}\left[\frac{2\rho\left(|\zeta_1|\ |\zeta_2|\right)^{1/2}}{h(1-\rho^2)}\right]\right\}$$

$$\cdot \exp\left[j\,(v_1\zeta_1 + v_2\zeta_2)\right] d\zeta_1\, d\zeta_2 \qquad \text{(A-2)}$$

At first glance, it appears impossible to obtain a closed form solution for Eq (A-2). However, after much trial and error, a valid solution was obtained.

The first step in solving Eq (A-2) is to remove the dependence of the integral upon the absolute value functions and the signum functions. This can be done by using the following relationships.

$$|\zeta_{1,2}| = -\zeta_{1,2} \quad \text{for} \quad -\pi < \zeta_{1,2} < 0$$

$$= \zeta_{1,2} \quad \text{for} \quad 0 < \zeta_{1,2} < \pi \qquad \text{(A-3)}$$

$$\mathrm{sgn}(\zeta_{1,2}) = -1 \quad \text{for} \quad -\pi < \zeta_{1,2} < 0$$

$$= +1 \quad \text{for} \quad 0 < \zeta_{1,2} < \pi \qquad \text{(A-4)}$$

Using equations (A-3) and (A-4) the above Fourier transform integral, Eq (A-2), can be written as the sum of four integrals.

$$\phi_2^E(v_1, v_2) = I_1 + I_2 + I_3 + I_4 \qquad \text{(A-5)}$$

The values of I_1 through I_4 are given on the next two pages. Also, a substitution of $c = h(1-\rho^2)$ is used for brevity.

$$I_1 = \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{1}{4hc} \exp\left[\frac{\zeta_1 + \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho\left(\zeta_1\zeta_2\right)^{1/2}}{c}\right] + \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{(2n+1)^2}\, I_{2n+1}\left[\frac{2\rho\left(\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right\} \cdot \exp\left[\, j\left(v_1\zeta_1 + v_2\zeta_2\right)\right] d\zeta_1\, d\zeta_2 \qquad \text{(A-6)}$$

$$I_2 = \int_{-\infty}^{0}\int_{0}^{\infty} \frac{1}{4hc} \exp\left[\frac{-\zeta_1 + \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho\left(-\zeta_1\zeta_2\right)^{1/2}}{c}\right] - \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{(2n+1)^2}\, I_{2n+1}\left[\frac{2\rho\left(-\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right\} \cdot \exp\left[\, j\left(v_1\zeta_1 + v_2\zeta_2\right)\right] d\zeta_1\, d\zeta_2 \qquad \text{(A-7)}$$

$$I_3 = \int_0^{\infty}\int_{-\infty}^{0} \frac{1}{4hc} \exp\left[\frac{\zeta_1 - \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho\left(-\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right.$$

$$\left. - \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{\left(2n + 1\right)^2} \quad I_{2n+1}\left[\frac{2\rho\left(-\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right\}$$

$$\cdot \exp\left\{ j\left(v_1\zeta_1 + v_2\zeta_2\right)\right\} d\zeta_1 \, d\zeta_2 \qquad \text{(A-8)}$$

$$I_4 = \int_0^{\infty}\int_{0}^{\infty} \frac{1}{4hc} \exp\left[\frac{-\zeta_1 - \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho\left(\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right.$$

$$\left. + \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{\left(2n + 1\right)^2} \quad I_{2n+1}\left[\frac{2\rho\left(\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right\}$$

$$\cdot \exp\left\{ j\left(v_1\zeta_1 + v_2\zeta_2\right)\right\} d\zeta_1 \, d\zeta_2 \qquad \text{(A-9)}$$

These integrals need to be arranged into a form that can be integrated or for which an integral is known. One form that works was found in [10:490] and is given by

$$\int_0^\infty x^{p-1} e^{-ax} \cos(mx)\, dx = \frac{\Gamma(p) \cos(p\theta)}{(a^2 + m^2)^{p/2}} \qquad (A-10)$$

$$\int_0^\infty x^{p-1} e^{-ax} \sin(mx)\, dx = \frac{\Gamma(p) \sin(p\theta)}{(a^2 + m^2)^{p/2}} \qquad (A-11)$$

$$\theta = \tan^{-1}\left[\frac{m}{a}\right] = \cos^{-1}\left[\frac{1}{(a^2 + m^2)^{1/2}}\right] \qquad (A-12)$$

Several manipulations must be performed to get integrals one through four into the above form. The first step is to make all the limits of integration to be from zero to infinity. This is done in two steps, first making all integrals start at zero, then changing any negative infinities to positive by changing the sign of the appropriate variable. These changes are shown on the next four pages.

Step 1: Change order of limits.

$$I_1 = \int_0^{-\infty}\int_0^{-\infty} \frac{1}{4hc} \exp\left[\frac{\zeta_1 + \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho\left(\zeta_1\zeta_2\right)^{1/2}}{c}\right] + \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{\left(2n+1\right)^2} I_{2n+1}\left[\frac{2\rho\left(\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right\} \cdot \exp\left[j\left(v_1\zeta_1 + v_2\zeta_2\right)\right] d\zeta_1\, d\zeta_2 \qquad \text{(A-13)}$$

$$I_2 = -\int_0^{-\infty}\int_0^{\infty} \frac{1}{4hc} \exp\left[\frac{-\zeta_1 + \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho\left(-\zeta_1\zeta_2\right)^{1/2}}{c}\right] - \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{\left(2n+1\right)^2} I_{2n+1}\left[\frac{2\rho\left(-\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right\} \cdot \exp\left[j\left(v_1\zeta_1 + v_2\zeta_2\right)\right] d\zeta_1\, d\zeta_2 \qquad \text{(A-14)}$$

$$I_3 = -\int_0^{\infty}\int_0^{-\infty} \frac{1}{4hc} \exp\left[\frac{\zeta_1 - \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho\left(-\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right.$$

$$\left. - \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{\left(2n+1\right)^2} \quad I_{2n+1}\left[\frac{2\rho\left(-\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right\}$$

$$\cdot \exp\left[j \left(v_1\zeta_1 + v_2\zeta_2\right)\right] d\zeta_1 \, d\zeta_2 \qquad \text{(A-15)}$$

$$I_4 = \int_0^{\infty}\int_0^{\infty} \frac{1}{4hc} \exp\left[\frac{-\zeta_1 - \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho\left(\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right.$$

$$\left. + \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{\left(2n+1\right)^2} \quad I_{2n+1}\left[\frac{2\rho\left(\zeta_1\zeta_2\right)^{1/2}}{c}\right]\right\}$$

$$\cdot \exp\left[j \left(v_1\zeta_1 + v_2\zeta_2\right)\right] d\zeta_1 \, d\zeta_2 \qquad \text{(A-16)}$$

Step 2: Make all limits positive.

$$I_1 = \int_0^\infty \int_0^\infty \frac{1}{4hc} \exp\left[\frac{-\zeta_1 - \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho(\zeta_1\zeta_2)^{1/2}}{c}\right]\right.$$

$$\left. + \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{(2n+1)^2} \quad I_{2n+1}\left[\frac{2\rho(\zeta_1\zeta_2)^{1/2}}{c}\right]\right\}$$

$$\cdot \exp\left[j \left(-v_1\zeta_1 - v_2\zeta_2\right)\right] d\zeta_1 \, d\zeta_2 \qquad \text{(A-17)}$$

$$I_2 = \int_0^\infty \int_0^\infty \frac{1}{4hc} \exp\left[\frac{-\zeta_1 - \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho(\zeta_1\zeta_2)^{1/2}}{c}\right]\right.$$

$$\left. - \frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{1}{(2n+1)^2} \quad I_{2n+1}\left[\frac{2\rho(\zeta_1\zeta_2)^{1/2}}{c}\right]\right\}$$

$$\cdot \exp\left[j \left(v_1\zeta_1 - v_2\zeta_2\right)\right] d\zeta_1 \, d\zeta_2 \qquad \text{(A-18)}$$

$$I_3 = \int_0^\infty \int_0^\infty \frac{1}{4hc} \exp\left[\frac{-\zeta_1 - \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho(\zeta_1\zeta_2)^{1/2}}{c}\right]\right.$$

$$\left. - \frac{8}{\pi^2} \sum_{n=o}^{\infty} \frac{1}{(2n+1)^2} \; I_{2n+1}\left[\frac{2\rho(\zeta_1\zeta_2)^{1/2}}{c}\right]\right\}$$

$$\cdot \exp\left\{ j\,(-v_1\zeta_1 + v_2\zeta_2)\right\} d\zeta_1\, d\zeta_2 \qquad \text{(A-19)}$$

$$I_4 = \int_0^\infty \int_0^\infty \frac{1}{4hc} \exp\left[\frac{-\zeta_1 - \zeta_2}{c}\right] \cdot \left\{ I_o\left[\frac{2\rho(\zeta_1\zeta_2)^{1/2}}{c}\right]\right.$$

$$\left. + \frac{8}{\pi^2} \sum_{n=o}^{\infty} \frac{1}{(2n+1)^2} \; I_{2n+1}\left[\frac{2\rho(\zeta_1\zeta_2)^{1/2}}{c}\right]\right\}$$

$$\cdot \exp\left\{ j\,(v_1\zeta_1 + v_2\zeta_2)\right\} d\zeta_1\, d\zeta_2 \qquad \text{(A-20)}$$

These integrals can be simplified by noting that they are really the sum of two integrands. Thus, if the simplifications listed below are made, the integrals can be written in a workable form.

$$A = I_o \left[\frac{2\rho(\zeta_1 \zeta_2)^{1/2}}{c} \right] \tag{A-21}$$

$$B = \frac{8}{\pi^2} \sum_{n=o}^{\infty} \frac{1}{(2n + 1)^2} I_{2n+1} \left[\frac{2\rho(\zeta_1 \zeta_2)^{1/2}}{c} \right] \tag{A-22}$$

$$D = \frac{1}{4hc} \exp \left[\frac{-\zeta_1 - \zeta_2}{c} \right] \tag{A-23}$$

With the use of these substitutions and equations (A-17) through (A-20), the integral Eq (A-5) becomes

$$\begin{aligned} \Gamma_2^{\varepsilon} = & \int_0^{\infty}\int_0^{\infty} D\,(A + B) \exp \left[j\left(-v_1\zeta_1 - v_2\zeta_2\right)\right] d\zeta_1 d\zeta_2 \\ & + \int_0^{\infty}\int_0^{\infty} D\,(A - B) \exp \left[j\left(-v_1\zeta_1 + v_2\zeta_2\right)\right] d\zeta_1 d\zeta_2 \\ & + \int_0^{\infty}\int_0^{\infty} D\,(A - B) \exp \left[j\left(+v_1\zeta_1 - v_2\zeta_2\right)\right] d\zeta_1 d\zeta_2 \\ & + \int_0^{\infty}\int_0^{\infty} D\,(A + B) \exp \left[j\left(+v_1\zeta_1 + v_2\zeta_2\right)\right] d\zeta_1 d\zeta_2 \end{aligned} \tag{A-24}$$

Grouping like terms, Eq (A-24) can be rewritten as

$$\tau_2^E = \int_0^\infty \int_0^\infty D\ (A + B) \left\{ \exp\left[j\left(v_1\zeta_1+v_2\zeta_2\right)\right] + \exp\left[-j\left(v_1\zeta_1+v_2\zeta_2\right)\right]\right\} d\zeta_1 d\zeta_2$$
$$+ \int_0^\infty \int_0^\infty D\ (A - B) \left\{ \exp\left[j\left(v_1\zeta_1-v_2\zeta_2\right)\right] + \exp\left[-j\left(v_1\zeta_1-v_2\zeta_2\right)\right]\right\} d\zeta_1 d\zeta_2 \quad \text{(A-25)}$$

Using Euler's identity $\exp(jx) + \exp(-jx) = 2\cos(x)$, Eq (A-25) can be written in the form

$$\tau_2^E = 2 \int_0^\infty \int_0^\infty D\ (A + B) \cos\left(v_1\zeta_1+v_2\zeta_2\right)$$
$$+ 2 \int_0^\infty \int_0^\infty D\ (A - B) \cos\left(v_1\zeta_1-v_2\zeta_2\right) \quad \text{(A-26)}$$

Equation (A-26) with the use of a the trigonometric identity for the addition and subtraction of angles, which states $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$, can be written as

$$\gamma_2^E = 2 \int_0^\infty \int_0^\infty D\,(A + B) \Big\{ \cos\left(v_1\zeta_1\right) \cos\left(v_2\zeta_2\right) - \sin\left(v_1\zeta_1\right) \sin\left(v_2\zeta_2\right) \Big\}$$
$$+ 2 \int_0^\infty \int_0^\infty D\,(A - B) \Big\{ \cos\left(v_1\zeta_1\right) \cos\left(v_2\zeta_2\right) + \sin\left(v_1\zeta_1\right) \sin\left(v_2\zeta_2\right) \Big\} \qquad \text{(A-27)}$$

Expansion of Eq (A-27) simplifies to

$$\gamma_2^E = 4 \int_0^\infty \int_0^\infty D \left[A \cos(v_1\zeta_1)\cos(v_2\zeta_2) \right] d\zeta_1 d\zeta_2$$
$$- 4 \int_0^\infty \int_0^\infty D \left[B \sin(v_1\zeta_1)\sin(v_2\zeta_2) \right] d\zeta_1 d\zeta_2 \qquad \text{(A-28)}$$

Now the following substitutions $\xi = \zeta_1/c$ and $\eta = \zeta_2/c$ are made and the Bessel function is replaced with its series equivalent Eq (A-29) found in [20:138].

$$I_\nu(z) = \left[\frac{z}{2}\right]^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\ \Gamma(\nu+k+1)} \qquad \text{(A-29)}$$

Since ν and k are integers, $\Gamma(\nu+k+1) = (\nu+k)!$ [20:101] and Eq (A-29) reduces to Eq (A-30).

$$I_\nu(z) = \left[\frac{z}{2}\right]^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\,(\nu+k)!} \tag{A-30}$$

Eq (A-30) and the ξ and η substitution are used to transform the variables A, B, and D, given in Eq (A-21) through (A-23), into the following equations.

$$A = I_o\,(2\rho\ \xi^{1/2}\eta^{1/2})$$

$$= \sum_{k=0}^{\infty} \frac{(\rho^2\xi\eta)^k}{k!^2} \tag{A-31}$$

$$B = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{I_{2n+1}(2\rho\ \xi^{1/2}\eta^{1/2})}{[2n+1]^2}$$

$$= \frac{8}{\pi^2} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\rho\ \xi^{1/2}\eta^{1/2})^{2n+1}\ (\rho^2\xi\eta)^k}{(2n+1)^2\ k!\ (2n+1+k)!} \tag{A-32}$$

$$D = \frac{1}{4hc} \exp\,(-\xi - \eta) \tag{A-33}$$

Substituting Eq (A-31) through (A-33) into Eq (A-28) allows the integral to be written as

$$\tau_2^E = \frac{c}{h} \sum_{k=0}^{\infty} \left\{ \frac{\rho^{2k}}{k!^2} \int_0^{\infty} \exp(-\xi)\ \xi^k \cos(cv_1\xi)\ d\xi \right.$$

$$\left. \times \int_0^{\infty} \exp(-\eta)\ \eta^k \cos(cv_2\eta)\ d\eta \right\}$$

$$- \frac{8c}{\pi^2 h} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\rho^{2n+1+2k}}{(2n+1)^2\ k!\ (2n+1+k)!}$$

$$\times \int_0^{\infty} \exp(-\xi)\ \xi^{n+k+1/2} \sin(cv_1\xi)\ d\xi$$

$$\times \int_0^{\infty} \exp(-\eta)\ \eta^{n+k+1/2} \sin(cv_2\eta)\ d\eta \qquad \text{(A-34)}$$

Eq (A-34) now has its integrals in the form of equations (A-10) and (A-11) for which solutions are known. Therefore, the characteristic function γ_2^E for Pyati's two-dimensional exponential pdf has the following solution.

$$\gamma_2^E = \frac{c}{h} \sum_{k=0}^{\infty} \frac{\rho^{2k} \quad \Gamma^2(k+1) \quad \cos\left[(k+1)\theta_1\right] \quad \cos\left[(k+1)\theta_2\right]}{k!^2 \quad \left[1 + (cv_1)^2\right]^{(k+1)/2} \quad \left[1 + (cv_2)^2\right]^{(k+1)/2}}$$

$$- \frac{8c}{\pi^2 h} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\rho^{2n+1+2k}}{(2n+1)^2 \; k! \; (2n+1+k)!}$$

$$\cdot \frac{\Gamma^2(n+k+1.5) \; \sin\left[(n+k+1.5)\theta_1\right] \; \sin\left[(n+k+1.5)\theta_2\right]}{\left[1+(cv_1)^2\right]^{(n+k+1.5)/2} \quad \left[1+(cv_2)^2\right]^{(n+k+1.5)/2}} \tag{A-35}$$

Where θ_1 and θ_2 are given as

$$\theta_{1,2} = \tan^{-1}\left[cv_{1,2} \right]$$

$$= \cos^{-1}\left[\frac{1}{(1 + c^2 v_{1,2}^2)^{1/2}} \right]$$

$$= \sin^{-1}\left[\frac{(1-\rho^2) h v_{1,2}}{(1 + c^2 v_{1,2}^2)^{1/2}} \right] \tag{A-36}$$

Finally, letting $v_1 = -v_2 = v$ the required form of the characteristic function ι_2^E is obtained.

$$\iota_2^E\,(v_z,-v_z;\rho) = (1-\rho^2)\sum_{k=0}^{\infty} \frac{\rho^{2k}\cos^2\left[(k+1)\theta\right]}{\left[1 + ((1-\rho^2)hv)^2\right]^{(k+1)}}$$

$$+ \frac{8}{\pi^2}\,(1-\rho^2)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\rho^{2n+1+2k}}{(2n+1)^2\,k!\,(2n+1+k)!}$$

$$\times \frac{\Gamma^2(n+k+1.5)\sin^2\left[(n+k+1.5)\theta\right]}{\left[1 + ((1-\rho^2)hv)^2\right]^{(n+k+1.5)}} \qquad \text{(A-37)}$$

$$\theta = \tan^{-1}\left[cv\right] = \cos^{-1}\left[\frac{1}{\left[1 + ((1-\rho^2)hv)^2\right]^{1/2}}\right]$$

$$= \sin^{-1}\left[\frac{(1-\rho^2)hv}{\left[1 + ((1-\rho^2)hv)^2\right]^{1/2}}\right] \qquad \text{(A-38)}$$

Appendix B

A Closed Form Solution for the First Series of the Characteristic Function

The purpose of this appendix is to show how a closed form solution for the first series of Eq (A-35) is derived. The first series of Eq (A-35) is repeated in this appendix as Eq (B-1). The derivation of how a closed form solution for this equation follows.

$$X_1 = (1-\rho^2)\sum_{k=0}^{\infty}\left\{ \frac{\rho^{2k}\cos((k+1)\theta_1)}{\left[1+((1-\rho^2)hv_1)^2\right]^{(k+1)/2}} \cdot \frac{\cos((k+1)\theta_2)}{\left[1+((1-\rho^2)hv_2)^2\right]^{(k+1)/2}} \right\} \tag{B-1}$$

$$\theta_1 = \tan^{-1}\left[(1-\rho^2)hv_1\right] = \cos^{-1}\left[\frac{1}{\left[1+\left[(1-\rho^2)hv_1\right]^2\right]^{1/2}}\right]$$

$$= \sin^{-1}\left[\frac{(1-\rho^2)hv_1}{\left[1+\left[(1-\rho^2)hv_1\right]^2\right]^{1/2}}\right] \tag{B-2}$$

To obtain the variable θ_2, just replace v_1 with v_2 in Eq (B-2). The rest of the equation is the same.

Using Euler's identity for a cosine, the cosine term, $\cos[(k+1)\theta_1]$, is written as Eq (B-3). The same is done for the $\cos[(k+1)\theta_2]$ term by replacing θ_1 with θ_2.

$$\cos((k+1)\theta_1) = \frac{\exp[j(k+1)\theta_1] + \exp[-j(k+1)\theta_1]}{2} \tag{B-3}$$

Also, to reduce Eq (B-1) to a workable form, the following substitutions are used.

$$A = \left[1 + [(1-\rho^2)hv_1]^2 \right]^{1/2} \tag{B-4}$$

$$B = \left[1 + [(1-\rho^2)hv_2]^2 \right]^{1/2} \tag{B-5}$$

The use of equations (B-3) through (B-5) allows equation (B-1) to be written as

$$X_n = \frac{(1-\rho^2)}{4AB} \sum_{k=0}^{\infty} \frac{\rho^{2k}}{A^k B^k} \left\{ \left[\exp[j(k+1)\theta_1] + \exp[-j(k+1)\theta_1]\right] \right.$$

$$\cdot \left[\exp[j(k+1)\theta_2] + \exp[-j(k+ \ldots \right. \tag{B-6}$$

Expanding Eq (B-6) by multiplying terms to remove the brackets yields

$$X_a = \frac{(1-\rho^2)}{4AB} \sum_{k=0}^{\infty} \frac{\rho^{2k}}{A^k B^k} \Big\{ \exp\left[j(\theta_1+\theta_2)\right] \exp\left[jk(\theta_1+\theta_1)\right]$$

$$+ \exp\left[j(\theta_1-\theta_2)\right] \exp\left[jk(\theta_1-\theta_2)\right]$$

$$+ \exp\left[-j(\theta_1-\theta_2)\right] \exp\left[-jk(\theta_1-\theta_2)\right]$$

$$+ \exp\left[-j(\theta_1+\theta_2)\right] \exp\left[-jk(\theta_1+\theta_2)\right] \Big\} \qquad \text{(B-7)}$$

Using the geometric series identity, $\sum_{k=0}^{\infty} X^k = \frac{1}{1-X}$ which is valid when $|X| < 1$, [20:107] equation (B-7) can be written as Eq (B-8). Equation (B-8) is now in a close form since the summation sign has been removed.

$$X_a = \frac{(1-\rho^2)}{4} \Bigg\{ \frac{\exp\left[j(\theta_1+\theta_2)\right]}{AB - \rho^2 \exp\left[j(\theta_1+\theta_2)\right]}$$

$$+ \frac{\exp\left[j(\theta_1-\theta_2)\right]}{AB - \rho^2 \exp\left[j(\theta_1-\theta_2)\right]} + \frac{\exp\left[-j(\theta_1-\theta_2)\right]}{AB - \rho^2 \exp\left[-j(\theta_1-\theta_2)\right]}$$

$$+ \frac{\exp\left[-j(\theta_1+\theta_2)\right]}{AB - \rho^2 \exp\left[-j(\theta_1+\theta_2)\right]} \Bigg\} \qquad \text{(B-8)}$$

Euler's identities $\exp(jx) = \cos(x) + j\sin(x)$ and $\exp(-jx) = \cos(x) - j\sin(x)$ are used with the substitutions of the terms $\mathbb{M} = \cos(\theta_1+\theta_2)$, $\mathbb{N} = \sin(\theta_1+\theta_2)$, $\mathbb{P} = \cos(\theta_1-\theta_2)$ and $\mathbb{S} = \sin(\theta_1-\theta_2)$ to obtain Eq (B-9).

$$X_a = \frac{(1-\rho^2)}{4} \left\{ \frac{\mathbb{M} + j\mathbb{N}}{\mathbb{A}\mathbb{B} - \rho^2\mathbb{M} - j\rho^2\mathbb{N}} + \frac{\mathbb{P} + j\mathbb{S}}{\mathbb{A}\mathbb{B} - \rho^2\mathbb{P} - j\rho^2\mathbb{S}} \right.$$

$$\left. + \frac{\mathbb{P} - j\mathbb{S}}{\mathbb{A}\mathbb{B} - \rho^2\mathbb{R} + j\rho^2\mathbb{S}} + \frac{\mathbb{M} - j\mathbb{N}}{\mathbb{A}\mathbb{B} - \rho^2\mathbb{M} + j\rho^2\mathbb{N}} \right\} \tag{B-9}$$

Manipulating Eq (B-9) by multiplying each term's numerator and denominator by the conjugate of its denominator yields

$$X_a = \frac{(1-\rho^2)}{4} \left\{ \frac{2\mathbb{A}\mathbb{B}\mathbb{M} - 2\rho^2\mathbb{M}^2 - 2\rho^2\mathbb{N}^2}{\mathbb{A}^2\mathbb{B}^2 - 2\mathbb{A}\mathbb{B}\rho^2\mathbb{M} + \rho^4\mathbb{M}^2 + \rho^4\mathbb{N}^2} \right.$$

$$\left. + \frac{2\mathbb{A}\mathbb{B}\mathbb{P} - 2\rho^2\mathbb{P}^2 - 2\rho^2\mathbb{S}^2}{\mathbb{A}^2\mathbb{B}^2 - 2\mathbb{A}\mathbb{B}\rho^2\mathbb{P} + \rho^4\mathbb{P}^2 + \rho^4\mathbb{S}^2} \right\} \tag{B-10}$$

Knowing that $\mathbb{M}^2 + \mathbb{N}^2 = \cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2) = 1$ and $\mathbb{P}^2 + \mathbb{S}^2 = \cos^2(\theta_1-\theta_2) + \sin^2(\theta_1-\theta_2) = 1$ allows Eq (B-10) to be reduced to Eq (B-11).

$$X_a = \frac{(1-\rho^2)}{2} \left\{ \frac{\mathbb{A}\mathbb{B}\mathbb{M} - \rho^2}{\mathbb{A}^2\mathbb{B}^2 - 2\mathbb{A}\mathbb{B}\rho^2\mathbb{M} + \rho^4} \right.$$

$$\left. + \frac{\mathbb{A}\mathbb{B}\mathbb{P} - \rho^2}{\mathbb{A}^2\mathbb{B}^2 - 2\mathbb{A}\mathbb{B}\rho^2\mathbb{P} + \rho^4} \right\} \qquad \text{(B-11)}$$

In Eq (B-12) and (B-13), the trigonometric identity for the cosine of the sum of two angles is used to expand $\mathbb{M}$ and $\mathbb{P}$ in terms of sine and cosine of θ. Also, Eq (B-2), Eq (B-4) and Eq (B-5) are used to evaluate the sine and cosine of θ.

$$\mathbb{M} = \cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \qquad \text{(B-12)}$$

$$\mathbb{P} = \cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 \qquad \text{(B-13)}$$

$$\cos\theta_1 = \cos(\cos^{-1}(1/\mathbb{A})) = 1/\mathbb{A} \qquad \text{(B-14)}$$

$$\cos\theta_2 = \cos(\cos^{-1}(1/\mathbb{B})) = 1/\mathbb{B} \qquad \text{(B-15)}$$

$$\sin\theta_1 = \sin\left[\sin^{-1}\left[\frac{(1-\rho^2)hv_1}{\mathbb{A}}\right]\right] = \frac{(1-\rho^2)hv_1}{\mathbb{A}} \qquad \text{(B-16)}$$

$$\sin\theta_2 = \sin\left[\sin^{-1}\left[\frac{(1-\rho^2)hv_2}{\mathbb{B}}\right]\right] = \frac{(1-\rho^2)hv_2}{\mathbb{B}} \qquad \text{(B-17)}$$

Equations (B-12) through (B-17) are used to write new equation for $\mathbb{M}$ and $\mathbb{P}$. These terms are now void of sine and cosine terms.

$$M = \frac{1 - (1-\rho^2)hv_1 \ (1-\rho^2)hv_2}{\mathbb{A}\mathbb{B}} = \frac{1 - (1-\rho^2)^2 h^2 v_1 v_2}{\mathbb{A}\mathbb{B}} \qquad \text{(B-18)}$$

$$\mathbb{P} = \frac{1 + (1-\rho^2)hv_1 \ (1-\rho^2)hv_2}{\mathbb{A}\mathbb{B}} = \frac{1 + (1-\rho^2)^2 h^2 v_1 v_2}{\mathbb{A}\mathbb{B}} \qquad \text{(B-19)}$$

Substituting equations (B-18) and (B-19) into Eq (B-11) yields Eq (B-20). The values for $\mathbb{A}^2\mathbb{B}^2$ are given in Eq (B-21).

$$X_a = \frac{(1-\rho^2)}{2}\left\{ \frac{1 - (1-\rho^2)^2 h^2 v_1 v_2 - \rho^2}{\mathbb{A}^2\mathbb{B}^2 - 2\rho^2\left[1-(1-\rho^2)^2 h^2 v_1 v_2\right] + \rho^4} \right.$$

$$\left. \frac{1 + (1-\rho^2)^2 h^2 v_1 v_2 - \rho^2}{\mathbb{A}^2\mathbb{B}^2 - 2\rho^2\left[1+(1-\rho^2)^2 h^2 v_1 v_2\right] + \rho^4} \right\} \qquad \text{(B-20)}$$

$$\mathbb{A}^2\mathbb{B}^2 = \left[1 + ((1-\rho^2)hv_1)^2\right]\left[1 + ((1-\rho^2)hv_2)^2\right] \qquad \text{(B-21)}$$

Combining Eq (B-20) and (B-21) and factoring like terms reduces the final equation to a close form solution for the first series of Eq (B-1).

$$X_a = \frac{1}{2}\left\{ \frac{1 - (1-\rho^2)h^2 v_1 v_2}{(1-\rho^2)^2 h^4 v_1^2 v_1^2 + h^2 v_1^2 + h^2 v_2^2 + 2\rho^2 h^2 v_1 v_2 + 1} \right.$$

$$\left. + \frac{1 + (1-\rho^2)h^2 v_1 v_2}{(1-\rho^2)^2 h^4 v_1^2 v_1^2 + h^2 v_1^2 + h^2 v_2^2 - 2\rho^2 h^2 v_1 v_2 + 1} \right\} \qquad \text{(B-22)}$$

When $v_1 = -v_2 = v$ is used in Eq (B-22), the equation can be written as

$$X_a = \frac{1}{2}\left\{ \frac{1+(1-\rho^2)h^2v^2}{(1-\rho^2)^2h^4v^4 + 2h^2v^2 - 2\rho^2h^2v^2 + 1} + \frac{1-(1-\rho^2)h^2v^2}{(1-\rho^2)^2h^4v^4 + 2h^2v^2 + 2\rho^2h^2v^2 + 1} \right\} \tag{B-23}$$

Equation (B-22) and (B-23) agrees with a solution obtained independently by Roseman [15]. Also, when $\rho = 0$ Eq (B-23) is equal to

$$X_a = \frac{1}{\left(h^2v^2 + 1\right)^2} \tag{B-24}$$

which is equal to the marginal of Pyati's exponential pdf squared as it should be.

Appendix C

Computer Program

To compute the shadowing function for Pyati's exponential pdf the following computer program was written. Figure 8 shows a flowchart of the program. The program is listed after the flowchart.

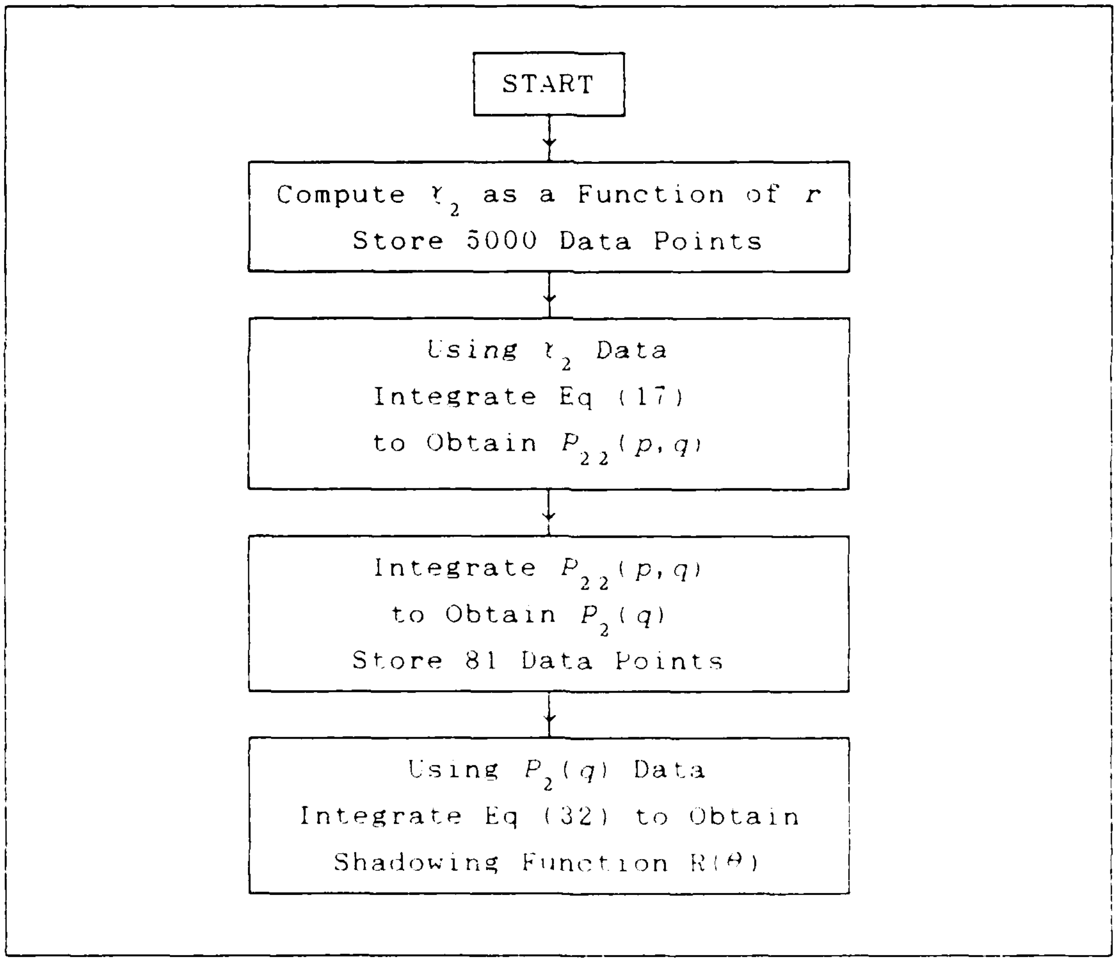

Figure 8. Program Flowchart

The following program was written to compute the shadowing for Pyati's exponential pdf.

```
'/////////////////////////////////////////////////////////////////
'//
'//                        COMPUTE τ2
'//
'/////////////////////////////////////////////////////////////////

'    Variables
'          corlen  -  correlation length
'               w  -  square root of the mean-square slope
'               k  -  wavenumber
'              pi  -  3.141592654
'               r  -  separation distance
'              pr  -  correlation coefficient
'            char  -  characteristic function's value
'               p  -  surface slope x direction
'               q  -  surface slope y direction
'              ro  -  shadowing function

DEFDBL A-Z  ' Double Precision used for all Variables
open "c:pam10.dat" for output as #1
     corlen = 1
     w = 1.41
     k = 10
     pi = 4*atn(1)
     v = k
     h = corlen*w/2
     hv = h*v
     dim qval(85),pval(85),value(5005),w(5005)
     dim p22(85,85),temp(85)
     for cnt = 0  to 5000
          r = cnt/1000
          pr = exp(-r*r/(corlen*corlen))
          call chf2(h,v,pr,char)
          w(cnt) = char
          print #1, cnt,r,char
     next cnt
close

'/////////////////////////////////////////////////////////////////
'//
'//                 COMPUTE  P22(p,q) and P2(q)
'//
'/////////////////////////////////////////////////////////////////

open "c:paj.dat" for output as #1
     for qq = 0 to 80
          q = qq/20
          for pp = qq to 80
```

```
                p = pp/20
                for cnt = 0 to 5000
                    r = cnt/1000
                    pq = sqr(p*p+q*q)
                    call bes10(k*r*pq,bes)
                    value(cnt) = r * bes * w(cnt)
                next cnt
                nump = 5001 : delta = .001
                call simpson(value(),nump,delta,result)
                p22(qq,pp) = v*v*result/(2*pi)
                p22(pp,qq) = p22(qq,pp)
                print q,p,p22(qq,pp)
            next pp
        next qq
        for qq = 0 to 80
            for pp = 0 to 80
                pval(pp) = p22(qq,pp)
            next pp
            nump = 81 :delta = .05
            call simpson(pval(),nump,delta,result)
            qval(qq) = 2*result
            print #1 ,qq,q,qval(qq)
            print q,qval(qq)
        next qq
close


'//////////////////////////////////////////////////////////////////
'//
'//            COMPUTE  SHADOWING FUNCTION  R(θ)
'//
'//////////////////////////////////////////////////////////////////

open "c:pab.dat" for output as #1
        for j = 1 to 80
            jj = j * .05
            for g = j to 80
                gg = g*.05
                cnt = 1+g-j
                temp(cnt) = ((gg/jj)-1)*qval(g)
            next g
            npoint = 80 - j
            delta = .05
            call simpson(temp(),npoint,delta,result)
            ro = 1/(1+ result )
            ang = 90-(57.2958*atn(jj))
            print #1, ang,ro
            print jj, ang,ro
        next j
close
end
```

```
'///////////////////////////////////////////////////////////////////
'//
'//
'//                  Subroutine that computes τ2
'//
'//
'///////////////////////////////////////////////////////////////////

sub chf2(h,v,pr,char) static
     pi = 4*atn(1)
     hv = h*v
     pterm = 1-pr*pr
     c = h*(1-pr^2)
     termhvp = pterm*hv*hv
     ang = atn(c*v)
     aterm1 = 1 + (c*v)^2
     char = 0
     term11 = (1-termhvp)/(termhvp^2+2*(1+p*p)*hv*hv+1)
     term12 = (1+termhvp)/(termhvp^2+2*termhvp+1)
     firstsum = (term11+term12)/2
     fu = pr*pi/(4*aterm1^1.5)
     sum2 = fu*sin(1.5*ang)*sin(1.5*ang)
     n = 1
nloopin:
     fu = (fu*pr*pr*(2*n+1)*(2*n+1)/(n*(n+1)*4*aterm1)
     tempx = fu
     tempy = tempx
     if (tempx < 3D-9 ) then goto nloopout
     a = 1
     j = n
     k = n+2
loopin:
     a2 = 2*a
     tempx = tempx*(a2-1)*(a2-1)*j/((a2+1)*(a2+1)*k)
     tempy = tempy + tempx
     j = j - 1
     k = k + 1
     a = a + 1
     if (j = 0  or a > 18) then goto loopout
     goto loopin
loopout:
     sum2 = sum2 + tempy*sin((n+1.5)*ang)*sin((n+1.5)*ang)
     n = n + 1
     goto nloopin
nloopout:
     secondsum = 8*c*sum2/(h*pi*pi)
     char = firstsum + secondsum
end sub
```

```
'//////////////////////////////////////////////////////////////
'//
'//        Subroutine that computes bessel functions
'//
'//////////////////////////////////////////////////////////////

sub bes10(x,j0) static
'
' Returns the value of the zeroth ordered bessel function
'
'Polynomial approximations taken from Abromowitz and
'Stegun pages 369-370
'
     j0 = 0 : theta = 0 : f0 = 0
     if (x < -3) then
          j1 = -99999   goto done
     end if
     dim ja(9),jb(9)
     if (x < 3 or x = 3) then
          goto m1
     else
          goto m2
     end if
m1:
     ja(0) =  1.00
     ja(1) = -2.2499997  : ja(2) = 1.2656208
     ja(3) = - .3163866  : ja(4) =  .044447900
     ja(5) = - .0039444  : ja(6) =  .0002100
     for n = 0 to 6
          j0 = j0 +ja(n)*(x/3)^(2*n)
     next n
     goto done
m2:
     ja(0) =  .79788456 : ja(1) = -.00000077
     ja(2) = -.00552740 : ja(3) = -.00009512
     ja(4) =  .00137237 : ja(5) = -.00072805
     ja(6) =  .00014476
     jb(0) = 3.00000000 : jb(1) = -.78539816
     jb(2) = -.04166397 : jb(3) = -.00003954
     jb(4) =  .00262573 : jb(5) = -.00054125
     jb(6) = -.00029333 : jb(7) =  .00013558
     for n = 0 to 7
          theta = theta + jb(n)*(3/x)^(n-1)
     next n
     for n = 0 to 6
          f0 = f0 + ja(n)*(3/x)^n
     next n
     j0 = x^(-1/2)*f0*cos(theta)
done:
     erase ja,jb
end sub
```

```
'//////////////////////////////////////////////////////////////
'//
'//                Simpson Integration Subroutine
'//
'//////////////////////////////////////////////////////////////

sub simpson(f(1),n,h,result) static
'Purpose:
'  Performs Simpson's rule of integration to a function defined
'  by a table of evenly spaced values.
'Parameters:
'  f() - Array of values of the function.
'  n   - Number of points
'  h   - Spacing between points
'  Results - Results
'
'Check to see if the number of panels is even
     npanel = n - 1
     nhalf = int(npanel/2)
     nbegin = 1
     result = 0
     if npanel - 2*nhalf = 0 then
         goto ev
     end if
' Number of panels is odd. Use 3/8 rule on first three
' panels and 1/3 rule on the rest of the panels.
     result = 3*h/8 * ( f(1) + 3*f(2) + 3*f(3) + f(4) )
     nbegin = 4
ev:
' apply 1/3 rule, add in first, second, and last values.
     result = result+h/3*(f(nbegin)+4*f(nbegin+1)+f(n))
     nbegin = nbegin + 2
     if nbegin = n then
         goto ext
     end if
' The pattern after nbegin+2 is repetitive.
' Get nend, the place to stop.
     nend = n-2
     for i = nbegin to nend step 2
         result = result + h/3 * (2*f(i) + 4*f(i+1))
     next i
ext:
end sub
```

Appendix D

Closed Form Shadowing Function for a Gaussian Probability Density Function

This appendix will demonstrate the development of the closed form solution of Brown's Shadowing function for a Gaussian pdf. This closed form solution of the Gaussian pdf was used to verify the numerical integration that was used to compute the shadowing function for Pyati's exponential pdf. The joint Gaussian probability density function, also call a bivariate Gaussian density, is written as [13:107]

$$P_G(\zeta_1,\zeta_2) = \frac{1}{2\pi h^2\left(1-\rho^2\right)^{1/2}} \exp\left[-\left(\frac{\zeta_1^2 - 2\zeta_1\zeta_2\rho + \zeta_2^2}{2h^2\left(1-\rho^2\right)} \right)\right] \qquad \text{(D-1)}$$

The characteristic function of a two-dimensional probability density function is defined as [5:189]

$$\Phi_2(v_1,v_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_2(\zeta_1,\zeta_2)\ \exp\left(jv_1\zeta_1 + jv_2\zeta_2\right)\ d\zeta_1 d\zeta_2 \qquad \text{(D-2)}$$

To find the characteristic function of the two-dimensional Gaussian pdf, replace $P_2(\zeta_1,\zeta_2)$ in Eq (D-2) with (D-1) and solve. The results of the integration is

$$\Phi_2(v_1,v_2) = \exp\left[-\frac{1}{2}\ h^2\left(v_1^2 + 2\rho v_1 v_2 + v_2^2\right)\right] \qquad \text{(D-3)}$$

Now let $v_1 = -v_2 = v$. Then $f_2(v,-v)$ can be written as

$$f_2(v,-v) = \exp\left[-h^2 v^2 \left(1-\rho\right)\right] \qquad \text{(D-4)}$$

The joint probability density function of the surface slopes is obtained when the Gaussian correlation coefficient, $\rho = \exp\left[-r^2/\ell^2\right]$, is substituted into Eq (D-4) which is then placed into Eq (17). Also, the marginal density f^2, which equals $\exp\left[-h^2 v^2\right]$ for the Gaussian correlation coefficient, approaches zero in the high frequency limit. The joint pdf of the surface slopes is written as

$$P_{22} = \frac{v^2}{2\pi} \int_0^\infty r\, J_0\left[r\, v \sqrt{\left(p^2 + q^2 \right)} \right]$$

$$\cdot \left\{ \exp\left[-h^2 v^2 \left[1 - \exp\left(-r^2/\ell^2 \right) \right] \right] \right\} dr \qquad \text{(D-5)}$$

Since only r close to zero contributes to the integral, the exponential term $\exp\left(-r^2/\ell^2\right)$ can be approximated by the first two terms of its Taylor series [illegible]. The first two terms of the series is equal to $1 - r^2/\ell^2$. Applying this approximation, Eq (D-5) becomes

$$P_{22} = \frac{v^2}{2\pi} \int_0^\infty r \, J_0\left[r \, v \sqrt{\left(p^2 + q^2 \right)} \right] \left\{ \exp\left[\frac{-h^2 v^2 r^2}{\ell^2} \right] \right\} dr \quad \text{(D-6)}$$

With a change of variable, $r = \sqrt{x}$, Eq (D-6) is in the form of the definite integral given below [20:148].

$$\int_0^\infty \exp(-ax) \, J_0\left(b\sqrt{x} \right) dx = \frac{\exp\left(b^2/4a \right)}{a} \quad \text{(D-7)}$$

By substituting w^2 for $4h^2/\ell^2$ and using Eq (D-7), the integration of Eq (D-6) results in the pdf of the surface slopes being given as

$$P_{22}(p,q) = \frac{1}{\pi w^2} \exp\left[- \left(\frac{p^2 + q^2}{w^2} \right) \right] \quad \text{(D-8)}$$

Equation (D-8) agrees with Barrick's solution for a Gaussian pdf [15:720-721]. Also, Barrick showed that w^2 is the mean-square slope of the rough surface [15:67].

Equation (D-[illegible]) is from the definition of how a marginal [illegible] from a [illegible] probability density. [illegible] equation was used to obtain the marginal probability density [illegible] of Eq (D-[illegible]), the [illegible] slope probability density function.

$$P_2(q) = \int_{-\infty}^{+\infty} P_{22}(p,q)\, dp \qquad \text{(D-9)}$$

The margin of Eq (D-8) is equal to

$$P_2(q) = \frac{1}{w\sqrt{\pi}} \exp\left[\frac{-q^2}{w^2}\right] \qquad \text{(D-10)}$$

where w^2 is the mean-square surface slope as defined by Barrick. q is the slope of the surface in the y direction.

Brown's shadowing function is given as [illegible]

$$[illegible] \qquad \text{(D-11)}$$

is defined by Brown as "the probability that a backscattering specular point of the surface will not be shadowed [illegible] is equal to [illegible], where [illegible] is the incidence angle measured from the vertical axis.

Inserting Eq (D-10) into (D-11) and then evaluating results in

$$[illegible] = [illegible] + \frac{1}{w\sqrt{\pi}} \left[\int [illegible] \exp\left(\frac{-[illegible]}{w^2}\right) [illegible] \right] \qquad \text{(D-12)}$$

Performing the two integrations, Eq (D-12) becomes

$$R(\theta) = \left\{ 1 + \frac{1}{w\sqrt{\pi}} \left[\frac{w^2}{2\mu} \exp\left[\frac{-\mu^2}{w^2} \right] - \frac{w\sqrt{\pi}}{2} \operatorname{erfc}\left[\frac{\mu}{w} \right] \right] \right\}^{-1} \quad \text{(D-13)}$$

which reduces to

$$R(\theta) = \left\{ 1 + \frac{w}{2\mu\sqrt{\pi}} \exp\left[\frac{-\mu^2}{w^2} \right] - \frac{1}{2} \operatorname{erfc}\left[\frac{\mu}{w} \right] \right\}^{-1} \quad \text{(D-14)}$$

Equation (D-14) is a closed form solution to Brown's shadowing function for a Gaussian pdf with $w^2 = 4h^2/\ell^2$. The shadowing function for Beckmann's Gaussian pdf can be obtained by substituting $2m^2$ for w^2 in Eq (D-14).

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The purpose of this study was to develop a shadowing function for a rough surface model that was based upon an exponential probability density function (pdf). This study had four critical parts:
(1) Derive the characteristic function and the joint slope-probability density function for a second-order exponential pdf. (2) Write a computer program to numerically compute the shadowing function for the second-order exponential pdf surface model. (3) Compare the shadowing function obtained with an 'exponential like' and Gaussian pdf. (4) Also, investigate a discrepancy in the mean-square total slope term.

The study found that the Shadowing function for the exponential pdf had a lower value than the shadowing function for the 'exponential like' and gaussian pdfs. The shadowing functions for the 'exponential like' and Gaussian pdf were found to have about the same value. This study also found that the shadowing function is sensitive to the value of the mean-square slope term used.

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